EXERCISE 3.3

Q.1 Evaluate the following determinants.

(i) $\begin{vmatrix} 5 & -2 & -4 \\ 3 & -1 & -3 \\ -2 & 1 & 2 \end{vmatrix}$ (ii) $\begin{vmatrix} 5 & 2 & -3 \\ 3 & -1 & 1 \\ -2 & 1 & -2 \end{vmatrix}$

(iii) $\begin{vmatrix} 1 & 2 & -3 \\ -1 & 3 & 4 \\ -2 & 5 & 6 \end{vmatrix}$ (iv) $\begin{vmatrix} a+l & a-l & a \\ a & a+l & a-l \\ a-l & a & a+l \end{vmatrix}$

(v) $\begin{vmatrix} 1 & 2 & -2 \\ -1 & 1 & -3 \\ 2 & 4 & -1 \end{vmatrix}$ (vi) $\begin{vmatrix} 2a & a & a \\ b & 2b & b \\ c & c & 2c \end{vmatrix}$

Solution:

(i) $\begin{vmatrix} 5 & -2 & -4 \\ 3 & -1 & -3 \\ -2 & 1 & 2 \end{vmatrix}$

Expanding the determinant by R_1.

$= 5\begin{vmatrix} -1 & -3 \\ 1 & 2 \end{vmatrix} - (-2)\begin{vmatrix} 3 & -3 \\ -2 & 2 \end{vmatrix} + (-4)\begin{vmatrix} 3 & -1 \\ -2 & 1 \end{vmatrix}$

$= 5\big(2 \times (-1) - 1 \times (-3)\big) + 2\big(3 \times 2 - (-3) \times (-2)\big) - 4\big(3 \times 1 - (-2) \times (-1)\big)$

$= 5(-2+3) + 2(6-6) - 4(3-2)$

$= 5(1) + 2(0) - 4(1)$

$= 5 + 0 - 4 = 1$

(ii) $\begin{vmatrix} 5 & 2 & -3 \\ 3 & -1 & 1 \\ -2 & 1 & -2 \end{vmatrix}$

Expanding the determinant by R_1.

$= 5\begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ -2 & 2 \end{vmatrix} + (-3)\begin{vmatrix} 3 & -1 \\ -2 & 1 \end{vmatrix}$

$= 5\big(2 \times (-1) - 1 \times (1)\big) - 2\big((3) \times (-2) - (-2) \times (1)\big) - 3\big((3) \times (1) - (-2) \times (-1)\big)$

$= 5(2-1) - 2(-6+2) - 3(3-2)$

$= 5(1) - 2(-4) - 3(1)$

$= 5 + 8 - 3 = 10$

(iii) $\begin{vmatrix} 1 & 2 & -3 \\ -1 & 3 & 4 \\ -2 & 5 & 6 \end{vmatrix}$

Expanding the given determinant by R_1

$= 1\begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} - 2\begin{vmatrix} -1 & 4 \\ -2 & 6 \end{vmatrix} + (-3)\begin{vmatrix} -1 & 3 \\ -2 & 5 \end{vmatrix}$

$= 1\left((3) \times (6) - (5) \times (4)\right) - 2\left((6) \times (-1) - (-2) \times (4)\right) - 3\left((-1) \times (5) - (-2) \times (3)\right)$

$= 1\,(18 - 20) - 2\,(-6 + 8) - 3\,(-5 + 6)$

$= -2 - 2\,(2) - 3\,(1)$

$= -2 - 4 - 3 = -9$

(iv) $\begin{vmatrix} a+l & a-l & a \\ a & a+l & a-l \\ a-l & a & a+l \end{vmatrix}$

expanding the given determinant by R_1

$= (a+l)\begin{vmatrix} a+l & a-l \\ a & a+l \end{vmatrix} - (a+l)\begin{vmatrix} a & a-l \\ a-l & a+l \end{vmatrix} + a\begin{vmatrix} a & a+l \\ a-l & a \end{vmatrix}$

$= (a+l)\,[(a+l)(a+l) - a(a-l)] - (a-l)\,[a(a+l) - (a-l)(a-l)]$
$\qquad + a\,[(a)(a) - (a-l)(a+l)]$

$= (a+l)\,[a^2 + al + la + l^2 - a^2 + al] - (a-l)\,[a^2 + al - (a^2 - al - al + l^2)]$
$\qquad + a\,[a^2 - (a^2 + al - al - l^2)]$

$= (a+l)\,[3al + l^2] - (a-l)\,[a^2 + al - (a^2 - 2al + l^2)] + a\,[a^2 - (a^2 - l^2)]$

$= (a+l)\,[3al + l^2] - (a-l)\,[a^2 + al - a^2 + 2al - l^2)] + a\,[a^2 - a^2 + l^2)]$

$= (a+l)\,[3al + l^2] - (a-l)\,[3al - l^2)] + a\,[l^2)]$

$= 3a^2l + al^2 + 3al^2 + l^3 - (3a^2l - al^2 - 3al^2 + l^3) + al^2$

$= 3a^2l + 4al^2 + l^3 - 3a^2l + al^2 + 3al^2 - l^3 + al^2$

$= 9al^2$

2nd Method

$\begin{vmatrix} a+l & a-l & a \\ a & a+l & a-l \\ a-l & a & a+l \end{vmatrix}$

$= \begin{vmatrix} a+l+a-l+a & a-l & a \\ a+a+l+a-l & a+l & a-l \\ a-l+a+a+l & a & a+l \end{vmatrix}$ by $C_1 + C_2 + C_3$

$$= \begin{vmatrix} 3a & a-l & a \\ 3a & a+l & a-l \\ 3a & a & a+l \end{vmatrix}$$

Take common 3a from C_1

$$= 3a\begin{vmatrix} 1 & a-l & a \\ 1 & a+l & a-l \\ 1 & a & a+l \end{vmatrix}$$

$$= 3a\begin{vmatrix} 1 & a-l & a \\ 0 & a+l-a+l & a-l-a \\ 0 & a-a+l & a+l-a \end{vmatrix} \qquad \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix}$$

$$= 3a\begin{vmatrix} 1 & a-l & a \\ 0 & 2l & -l \\ 0 & l & l \end{vmatrix}$$

expanding by C_1

$$= 3a\left[1\begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} - 0\begin{vmatrix} -1 & 4 \\ -2 & 6 \end{vmatrix} + 0\begin{vmatrix} -1 & 3 \\ -2 & 5 \end{vmatrix}\right]$$

$= 3a\,[(2l)\,(l) - (l)\,(-l)] - 0 + 0$

$= 3a\,[2l^2 + l^2]$

$= 3a\,(3l^2)$

$= 9al^2$

(v) $\begin{vmatrix} \mathbf{1} & \mathbf{2} & \mathbf{-2} \\ \mathbf{-1} & \mathbf{1} & \mathbf{-3} \\ \mathbf{2} & \mathbf{4} & \mathbf{-1} \end{vmatrix}$

Expanding the given determinant by R_1

$$= 1\begin{vmatrix} 1 & -3 \\ 4 & -1 \end{vmatrix} - 2\begin{vmatrix} -1 & -3 \\ 2 & -1 \end{vmatrix} + (-2)\begin{vmatrix} -1 & 1 \\ 2 & 4 \end{vmatrix}$$

$= \big((1)\,(-1) - (4)\,(-3)\big) - 2\,\big((-1)\,(-1) - (2)\,(-3)\big) - 2\,\big((-1)\,(4) - (2)\,(1)\big)$

$= (-1 + 12) - 2\,(1 + 6) - 2\,(-4 - 2)$

$= 11 - 2\,(7) - 2\,(-6)$

$= 11 - 14 + 12 = 9$

(vi) $\begin{vmatrix} \mathbf{2a} & \mathbf{a} & \mathbf{a} \\ \mathbf{b} & \mathbf{2b} & \mathbf{b} \\ \mathbf{c} & \mathbf{c} & \mathbf{2c} \end{vmatrix}$

Expanding the given determinant by R_1

$$= 2a\begin{vmatrix} 2b & b \\ c & 2c \end{vmatrix} - a\begin{vmatrix} b & b \\ c & 2c \end{vmatrix} + a\begin{vmatrix} b & 2b \\ c & c \end{vmatrix}$$

$= 2a\,\big((2b)\,(2c) - (b)\,(c)\big) - a\,\big((b)\,(2c) - (b)\,(c)\big) + a\,\big((b)\,(c) - (2b)\,(c)\big)$

$= 2a\,(4bc - bc) - a\,(2bc - bc) + a\,(bc - 2bc)$

$= 6abc - abc - abc = 4abc$

Q.2 Without expansion show that.

(i) $\begin{vmatrix} 6 & 7 & 8 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{vmatrix} = 0$ **(ii)** $\begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{vmatrix} = 0$

Solution:

L.H.S.

$$\begin{vmatrix} 6 & 7 & 8 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} 6 & 7-6 & 8 \\ 3 & 4-3 & 5 \\ 2 & 3-2 & 4 \end{vmatrix} = \begin{vmatrix} 6 & 1 & 8 \\ 3 & 1 & 5 \\ 2 & 1 & 4 \end{vmatrix} \quad C_2 - C_1$$

$$\begin{vmatrix} 6 & 1 & 8-6 \\ 3 & 1 & 5-3 \\ 2 & 1 & 4-2 \end{vmatrix} = \begin{vmatrix} 6 & 1 & 2 \\ 3 & 1 & 2 \\ 2 & 1 & 2 \end{vmatrix} \quad C_3 - C_1$$

Take common (2) from c_3

$$= 2\begin{vmatrix} 6 & 1 & 1 \\ 3 & 1 & 1 \\ 2 & 1 & 1 \end{vmatrix}$$

As C_2 and C_3 are same so determinant will be zero.

$$= 2(0) = 0 = \text{R.H.S.}$$

(ii) $\begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{vmatrix} = 0$

$$\text{L.H.S.} = \begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 3-1 & -1 \\ 1 & 1+0 & 0 \\ 2 & -3+5 & 5 \end{vmatrix} \quad C_2 + C_3$$

$$= \begin{vmatrix} 2 & 2 & -1 \\ 1 & 1 & 0 \\ 2 & 2 & 5 \end{vmatrix} = 0 = \text{R.H.S.}$$

As C_1 and C_2 are same so determinant will be zero.

(iii) $\begin{vmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{4} & \mathbf{5} & \mathbf{6} \\ \mathbf{7} & \mathbf{8} & \mathbf{9} \end{vmatrix} = \mathbf{0}$

L.H.S.

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2-1 & 3-1 \\ 4 & 5-4 & 6-4 \\ 7 & 8-7 & 9-7 \end{vmatrix} \qquad \begin{matrix} C_2 - C_1 \\ C_3 - C_1 \end{matrix}$$

$$= \begin{vmatrix} 1 & 1 & 2 \\ 4 & 1 & 2 \\ 7 & 1 & 2 \end{vmatrix}$$

Take (2) common from C_3

$$= (2)\begin{vmatrix} 1 & 1 & 1 \\ 4 & 1 & 1 \\ 7 & 1 & 1 \end{vmatrix} = 0 = \text{R.H.S.}$$

As C_2 and C_3 are same so determinant will be zero.

Q.3 Show that

(i) $\begin{vmatrix} a_{11} & a_{12} & a_{13}+\alpha_{13} \\ a_{21} & a_{22} & a_{23}+\alpha_{23} \\ a_{31} & a_{32} & a_{33}+\alpha_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \alpha_{13} \\ a_{21} & a_{22} & \alpha_{23} \\ a_{31} & a_{32} & \alpha_{33} \end{vmatrix}$

(ii) $\begin{vmatrix} 2 & 3 & 0 \\ 3 & 9 & 6 \\ 2 & 15 & 1 \end{vmatrix} = 9\begin{vmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ 2 & 5 & 1 \end{vmatrix}$

(iii) $\begin{vmatrix} a+l & a & a \\ a & a+l & a \\ a & a & a+l \end{vmatrix} = l^2(3a+l)$

(iv) $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & zx & xy \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$

(v) $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$

(vi) $\begin{vmatrix} b & -1 & a \\ a & b & 0 \\ 1 & a & b \end{vmatrix} = a^3 + b^3$

(vii) $\begin{vmatrix} r\cos\phi & 1 & -\sin\phi \\ 0 & 1 & 0 \\ r\sin\phi & 0 & \cos\phi \end{vmatrix} = r$

Solution:

(i) $$\begin{vmatrix} a_{11} & a_{12} & a_{13}+\alpha_{13} \\ a_{21} & a_{22} & a_{23}+\alpha_{23} \\ a_{31} & a_{32} & a_{33}+\alpha_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \alpha_{13} \\ a_{21} & a_{22} & \alpha_{23} \\ a_{31} & a_{32} & \alpha_{33} \end{vmatrix}$$

$$\text{L.H.S.} = \begin{vmatrix} a_{11} & a_{12} & a_{13}+\alpha_{13} \\ a_{21} & a_{22} & a_{23}+\alpha_{23} \\ a_{31} & a_{32} & a_{33}+\alpha_{33} \end{vmatrix}$$

Expanding by R_1

$$= a_{11}\begin{vmatrix} a_{22} & a_{23}+\alpha_{23} \\ a_{32} & a_{33}+\alpha_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23}+\alpha_{23} \\ a_{31} & a_{33}+\alpha_{33} \end{vmatrix} + (a_{13}+\alpha_{13})\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}[a_{22}(a_{33}+\alpha_{33}) - a_{32}(a_{23}+\alpha_{23})] - a_{12}[a_{21}(a_{33}+\alpha_{33}) - a_{31}(a_{23}+\alpha_{23})]$$
$$+ (a_{13}+\alpha_{13})[a_{21}a_{32} - a_{31}a_{22}]$$

$$= a_{11}[a_{22}a_{33} + a_{22}\alpha_{33} - a_{32}a_{23} - a_{32}\alpha_{23}]$$
$$- a_{12}[a_{21}a_{33} + a_{21}\alpha_{33} - a_{31}a_{23} - a_{31}\alpha_{23}]$$
$$+ a_{13}[a_{21}a_{32} - a_{31}a_{22}] + \alpha_{13}[a_{21}a_{32} - a_{31}a_{22}] \quad \text{......... (1)}$$

Now take R.H.S.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \alpha_{13} \\ a_{21} & a_{22} & \alpha_{23} \\ a_{31} & a_{32} & \alpha_{33} \end{vmatrix}$$

Expanding both by R_1

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$
$$+ a_{11}\begin{vmatrix} a_{22} & \alpha_{23} \\ a_{32} & \alpha_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & \alpha_{23} \\ a_{31} & \alpha_{33} \end{vmatrix} + \alpha_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}[a_{22}a_{33} - a_{32}a_{23}] - a_{12}[a_{21}a_{33} - a_{23}a_{31}] + a_{13}[a_{21}a_{32} - a_{22}a_{31}]$$
$$+ a_{11}[a_{22}\alpha_{33} - a_{32}\alpha_{23}] - a_{12}[a_{21}\alpha_{33} - \alpha_{23}a_{31}] + \alpha_{13}[a_{21}a_{32} - a_{22}a_{31}]$$

$$= a_{11}[a_{22}a_{33} + a_{22}\alpha_{33} - a_{32}a_{23} - a_{32}\alpha_{23}] - a_{12}[a_{21}a_{33} + a_{21}\alpha_{33} - a_{23}a_{31} - \alpha_{23}a_{31}]$$
$$+ a_{13}[a_{21}a_{32} - a_{22}a_{31}] + \alpha_{13}[a_{21}a_{32} - a_{22}a_{31}] \quad \text{......... (2)}$$

from equation (1) and (2) L.H.S. = R.H.S.

(ii) $\begin{vmatrix} 2 & 3 & 0 \\ 3 & 9 & 6 \\ 2 & 15 & 1 \end{vmatrix} = 9\begin{vmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ 2 & 5 & 1 \end{vmatrix}$

$$\text{L.H.S.} = \begin{bmatrix} 2 & 3 & 0 \\ 3 & 9 & 6 \\ 2 & 15 & 1 \end{bmatrix}$$

Take common (3) from C_2

$$= 3\begin{bmatrix} 2 & 1 & 0 \\ 3 & 3 & 6 \\ 2 & 5 & 1 \end{bmatrix}$$

Take common (3) from R_2

$$= 3 \times 3\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ 2 & 5 & 1 \end{bmatrix}$$

$$= 9\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ 2 & 5 & 1 \end{bmatrix}$$

$$= \text{R.H.S.}$$

(iii) $\begin{vmatrix} a+l & a & a \\ a & a+l & a \\ a & a & a+l \end{vmatrix} = l^2(3a+l)$

$$\text{L.H.S.} = \begin{vmatrix} a+l & a & a \\ a & a+l & a \\ a & a & a+l \end{vmatrix}$$

$$= \begin{vmatrix} a+l+a+a & a+a+l+a & a+a+a+l \\ a & a+l & a \\ a & a & a+l \end{vmatrix} \quad R_1 + R_2 + R_3$$

$$= \begin{vmatrix} 3a+l & 3a+l & 3a+l \\ a & a+l & a \\ a & a & a+l \end{vmatrix}$$

$(3a + l)$ common from R_1

$$= (3a+l)\begin{vmatrix} 1 & 1 & 1 \\ a & a+l & a \\ a & a & a+l \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 1-1 \\ a & a+l-a & a-a \\ a & a-a & a+l-a \end{vmatrix} \quad \begin{matrix} C_2 - C_1 \\ C_3 - C_1 \end{matrix}$$

$$= (3a + l)\begin{vmatrix} 1 & 0 & 0 \\ a & l & 0 \\ a & 0 & l \end{vmatrix}$$

Expanding by R_1

$$= (3a + l)\left[1\begin{vmatrix} l & 0 \\ 0 & l \end{vmatrix} - 0\begin{vmatrix} a & 0 \\ a & l \end{vmatrix} + 0\begin{vmatrix} a & l \\ a & 0 \end{vmatrix}\right]$$

$$= (3a + l)\,[1\,(l^2 - 0) - 0 + 0]$$

$$= (3a + l)\,(l^2)$$

$$= l^2\,(3a + l) = \text{R.H.S.}$$

(iv) $$\begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{x} & \mathbf{y} & \mathbf{z} \\ \mathbf{yz} & \mathbf{zx} & \mathbf{xy} \end{vmatrix} = \begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{x} & \mathbf{y} & \mathbf{z} \\ \mathbf{x^2} & \mathbf{y^2} & \mathbf{z^2} \end{vmatrix}$$

Solution:

$$\text{L.H.S.} = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & zx & xy \end{vmatrix}$$

Multiply C_1 by x, C_2 by y and C_3 by z

$$= \frac{1}{xyz}\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ xyz & xyz & xyz \end{vmatrix}$$

Take (xyz) common from R_3.

$$= \frac{xyz}{xyz}\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix}$$

Interchange R_1 and R_3.

$$= -\begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x & y & z \end{vmatrix}$$

$$= (-1)(-1)\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = \text{R.H.S.}$$

(v) $\begin{vmatrix} \mathbf{b+c} & \mathbf{a} & \mathbf{a} \\ \mathbf{b} & \mathbf{c+a} & \mathbf{b} \\ \mathbf{c} & \mathbf{c} & \mathbf{a+b} \end{vmatrix} = \mathbf{4abc}$

$$\text{L.H.S.} = \begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

$$= \begin{vmatrix} b+c & a-a & a \\ b & c+a-b & b \\ c & c-a-b & a+b \end{vmatrix} \quad C_2 - C_3$$

$$= \begin{vmatrix} b+c & 0 & a \\ b & c+a-b & b \\ c & c-a-b & a+b \end{vmatrix}$$

expanding by R_1

$$= (b+c)\begin{vmatrix} c+a-b & b \\ c-a-b & a+b \end{vmatrix} - 0 + a\begin{vmatrix} b & c+a-b \\ c & c-a-b \end{vmatrix}$$

$$= (b+c)\,[(a+b)(c+a-b) - b(c-a-b)] + a\,[b(c-a-b) - c(c+a-b)]$$

$$= (b+c)\,[ca + a^2 - ab + bc + ba - b^2 - bc + ba + b^2] + a\,[bc - ba - b^2 - c^2 - ac + bc]$$

$$= (b+c)\,[ac + a^2 + ab] + a\,[2bc - ba - b^2 - c^2 - ac]$$

$$= bac + ba^2 + ab^2 + ac^2 + a^2c + abc + 2abc - ba^2 - ab^2 - ac^2 - a^2c$$

$$= abc + abc + 2abc$$

$$= 4abc = \text{R.H.S.}$$

(vi) $\begin{vmatrix} \mathbf{b} & \mathbf{-1} & \mathbf{a} \\ \mathbf{a} & \mathbf{b} & \mathbf{0} \\ \mathbf{1} & \mathbf{a} & \mathbf{b} \end{vmatrix} = \mathbf{a^3 + b^3}$

expanding this determinant by R_1

$$= b\begin{vmatrix} b & 0 \\ a & b \end{vmatrix} - (-1)\begin{vmatrix} a & 0 \\ 1 & b \end{vmatrix} + a\begin{vmatrix} a & b \\ 1 & a \end{vmatrix}$$

$$= b(b^2 - 0) + 1(ab - 0) + a(a^2 - b)$$

$$= b(b^2) + ab + a(a^2 - b)$$

$$= b^3 + ab + a^3 - ab$$

$$= a^3 + b^3$$

$$= \text{R.H.S.}$$

(vii) $\begin{vmatrix} r\cos\phi & 1 & -\sin\phi \\ 0 & 1 & 0 \\ r\sin\phi & 0 & \cos\phi \end{vmatrix} = r$

expanding this determinant by R_1

$= r\cos\phi \begin{bmatrix} 1 & 0 \\ 0 & \cos\phi \end{bmatrix} - 1 \begin{bmatrix} 0 & 0 \\ r\sin\phi & \cos\phi \end{bmatrix} - \sin\phi \begin{bmatrix} 0 & 1 \\ r\sin\phi & 0 \end{bmatrix}$

$= r\cos\phi(\cos\phi - 0) - 1(0 - 0) - \sin\phi(0 - r\sin\phi)$

$= r\cos^2\phi + r\sin^2\phi$

$= r(\cos^2\phi + \sin^2\phi)$

$= r(1)$

$= r = \text{R.H.S.}$

(viii) $\begin{vmatrix} a & b+c & a+b \\ b & c+a & b+c \\ c & a+b & c+a \end{vmatrix} = a^3 + b^3 + c^3 - 3abc$

L.H.S. $= \begin{vmatrix} a & b+c & a+b \\ b & c+a & b+c \\ c & a+b & c+a \end{vmatrix}$

$= \begin{vmatrix} a+b+c & b+c & a+b \\ b+c+a & c+a & b+c \\ c+a+b & a+b & c+a \end{vmatrix} \quad C_1 + C_2$

Take $(a+b+c)$ common from c_1

$= (a+b+c) \begin{vmatrix} 1 & b+c & a+b \\ 1 & c+a & b+c \\ 1 & a+b & c+a \end{vmatrix}$

$= (a+b+c) \begin{vmatrix} 1 & b+c & a+b \\ 0 & c+a-b-c & b+c-a-b \\ 0 & a+b-b-c & c+a-a-b \end{vmatrix} \quad \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix}$

$= (a+b+c) \begin{vmatrix} 1 & b+c & a+b \\ 0 & a-b & c-a \\ 0 & a-c & c-b \end{vmatrix}$

Expanding by C_1

$= (a+b+c) \cdot 1 \cdot \begin{vmatrix} a-b & c-a \\ a-c & c-b \end{vmatrix} - 0 + 0$

$= (a+b+c)[(a-b)(c-b) - (c-a)(a-c)]$

$= (a+b+c)[ac - ab - bc + b^2 - ac + c^2 + a^2 - ac]$

$= (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ac)$

$= a^3 + b^3 + c^3 - 3abc$

$= \text{R.H.S.}$

(ix) $\begin{vmatrix} a+\lambda & b & c \\ a & b+\lambda & c \\ a & b & c+\lambda \end{vmatrix} = \lambda^2 (a+b+c+\lambda)$

$$\text{L.H.S.} = \begin{vmatrix} a+\lambda & b & c \\ a & b+\lambda & c \\ a & b & c+\lambda \end{vmatrix}$$

$$= \begin{vmatrix} a+b+\lambda+c & b & c \\ a+b+\lambda+c & b+\lambda & c \\ a+b+c+\lambda & b & c+\lambda \end{vmatrix} \quad C_1 + (C_2 + C_3)$$

Taking $(a+b+c+\lambda)$ common from C_1

$$= (a+b+c+\lambda) \begin{vmatrix} 1 & b & c \\ 1 & b+\lambda & c \\ 1 & b & c+\lambda \end{vmatrix}$$

$$= (a+b+c+\lambda) \begin{vmatrix} 1 & b & c \\ 0 & b+\lambda-b & c-c \\ 0 & b-b & c+\lambda-c \end{vmatrix} \quad \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix}$$

$$= (a+b+c+\lambda) \begin{vmatrix} 1 & b & c \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{vmatrix}$$

Expanding by C_1

$$= (a+b+c+\lambda)\left[1 \begin{vmatrix} \lambda & 0 \\ 0 & \lambda \end{vmatrix} - 0 + 0 \right]$$

$$= (a+b+c+\lambda)(\lambda^2 - 0)$$

$$= \lambda^2 (a+b+c+\lambda)$$

$$= \text{R.H.S.}$$

(x) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

$$\text{L.H.S.} = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix} \quad \begin{matrix} C_2 - C_1 \\ C_3 - C_1 \end{matrix}$$

Expanding by R_1

$$= 1 \begin{vmatrix} b-a & c-a \\ b^2-a^2 & c^2-a^2 \end{vmatrix} - 0 + 0$$

$$= (b-a)(c^2-a^2) - (c-a)(b^2-a^2)$$

$$= (b-a)(c-a)(c+a) - (c-a)(b-a)(b+a)$$

$= (b-a)(c-a)[c+a-(b+a)]$

$= (b-a)(c-a)(c+a-b-a)$

$= (b-a)(c-a)(c-b)$

$= -(a-b)(c-a)(-1)(b-c)$

$= (a-b)(b-c)(c-a)$

= R.H.S.

(xi) $\begin{vmatrix} \mathbf{b+c} & \mathbf{a} & \mathbf{a^2} \\ \mathbf{c+a} & \mathbf{b} & \mathbf{b^2} \\ \mathbf{a+b} & \mathbf{c} & \mathbf{c^2} \end{vmatrix} = \mathbf{(a+b+c)(a-b)(b-c)(c-a)}$

$$\text{L.H.S.} = \begin{vmatrix} b+c & a & a^2 \\ c+a & b & b^2 \\ a+b & c & c^2 \end{vmatrix}$$

$$= \begin{vmatrix} b+c+a & a & a^2 \\ c+a+b & b & b^2 \\ a+b+c & c & c^2 \end{vmatrix} \qquad C_1 + C_2$$

Take $(a+b+c)$ common from c_1

$$= (a+b+c)\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & a & a^2 \\ 1-1 & b-a & b^2-a^2 \\ 1-1 & c-a & c^2-a^2 \end{vmatrix} \qquad \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & (b-a)(b+a) \\ 0 & c-a & (c-a)(c+a) \end{vmatrix}$$

Take common $(b-a)$ from R_2, and $(c-a)$ from R_3

$$= (a+b+c)(b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix}$$

Expanding by C_1

$$= (a+b+c)(b-a)(c-a)\left[1\begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix} - 0\begin{vmatrix} a & a^2 \\ 1 & c+a \end{vmatrix} + 0\begin{vmatrix} a & a^2 \\ 1 & b+a \end{vmatrix}\right]$$

$= (a+b+c)(b-a)(c-a)[c+a-(b+a)]$

$= (a+b+c)(b-a)(c-a)(c+a-b-a)$

$= (a+b+c)(b-a)(c-a)(c-b)$

$= -(a+b+c)(a-b)(c-a)(c-b)$

$= (-1)(-1)(a+b+c)(a-b)(b-c)(c-a)$

$= (a+b+c)(a-b)(b-c)(c-a)$

$=$ R.H.S.

Q.4 (i) **If** $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 1 \end{bmatrix}$

Find A_{12}, A_{22}, A_{32}, and $|A|$

(ii) $B = \begin{bmatrix} 5 & -2 & 5 \\ 3 & -1 & 4 \\ -2 & 1 & -2 \end{bmatrix}$ **then**

Find B_{21}, B_{22}, B_{23} and $|B|$

Solution:

(i) $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 1 \end{bmatrix}$

$A_{12} = (-1)^{1+2} M_{12}$ where $M_{12} = \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix}$

$= (-1)^3 (0)$ $\qquad = (0)(1) - (-2)(0)$

$= 0$ $\qquad = 0$

$A_{22} = (-1)^{2+2} M_{22}$ where $M_{22} = \begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix}$

$= (-1)^4 (-5)$ $\qquad = (1)(1) - (-2)(-3)$

$= (1)(-5)$ $\qquad = 1 - 6 = -5$

$= -5$ $\qquad = -5$

$A_{32} = (-1)^{3+2} M_{32}$ where $M_{32} = \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$

$= (-1)^5 (0)$ $\qquad = (1)(0) - (-3)(0)$

$= 0$ $\qquad = 0$

Now

$$|A| = \begin{vmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 1 \end{vmatrix}$$

Expanding by R_1

$$= 1\begin{vmatrix} -2 & 0 \\ -2 & 1 \end{vmatrix} - 2\begin{vmatrix} 0 & 0 \\ -2 & 1 \end{vmatrix} + (-3)\begin{vmatrix} 0 & -2 \\ -2 & -2 \end{vmatrix}$$

$$= 1\left((-2)(1) - (-2)(0)\right) - 2\left((0)(1) - (-2)(0)\right) - 3\left((0)(-2) - (-2)(-2)\right)$$

$$= (-2-0) - 2(0-0) - 3(0-4)$$

$$= -2 - 0 - 3(-4)$$

$$= -2 + 12 = 10$$

(ii) $\mathbf{B} = \begin{bmatrix} \mathbf{5} & \mathbf{-2} & \mathbf{5} \\ \mathbf{3} & \mathbf{-1} & \mathbf{4} \\ \mathbf{-2} & \mathbf{1} & \mathbf{-2} \end{bmatrix}$

$B_{21} = (-1)^{2+1} M_{21}$ where $M_{21} = \begin{bmatrix} -2 & 5 \\ 1 & -2 \end{bmatrix}$

$= (-1)^3 (-1)$ $\qquad = (-2)(-2) - (1)(5)$

$= (-1)(-1) = 1$ $\qquad = 4 - 5 = -1$

$B_{22} = (-1)^{2+2} M_{22}$ where $M_{22} = \begin{bmatrix} 5 & 5 \\ -2 & -2 \end{bmatrix}$

$= (-1)^4 (0)$ $\qquad = (5)(-2) - (5)(-2)$

$= 0$ $\qquad = -10 + 10 = 0$

$B_{23} = (-1)^{2+3} M_{23}$ where $M_{23} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$

$= (-1)^5 (1)$ $\qquad = (5)(1) - (-2)(2)$

$= (-1)(1)$ $\qquad = 5 - 4$

$= -1$ $\qquad = 1$

Now $|B| = \begin{vmatrix} 5 & -2 & 5 \\ 3 & -1 & 4 \\ -2 & 1 & -2 \end{vmatrix}$

Expanding by R_1

$$= 5\begin{vmatrix} -1 & 4 \\ 1 & -2 \end{vmatrix} - (-2)\begin{vmatrix} 3 & 4 \\ -2 & -2 \end{vmatrix} + 5\begin{vmatrix} 3 & -1 \\ -2 & -1 \end{vmatrix}$$

$$= 5\left((-1)(-2) - (1)(4)\right) + 2\left((3)(-2) - (-2)(4)\right) + 5\left((3)(1) - (-1)(-2)\right)$$

$$= 5(2-4) + 2(-6+8) + 5(3-2)$$

$$= 5(-2) + 2(2) + 5(1)$$

$$= -10 + 4 + 5$$

$$= -1$$

Q.5 Without expansion verify that

(i) $\begin{vmatrix} \alpha & \beta+\gamma & 1 \\ \beta & \gamma+\alpha & 1 \\ \gamma & \alpha+\beta & 1 \end{vmatrix} = 0$

(ii) $\begin{vmatrix} 1 & 2 & 3x \\ 2 & 3 & 6x \\ 3 & 5 & 9x \end{vmatrix} = 0$

(iii) $\begin{vmatrix} 1 & a^2 & \frac{a}{bc} \\ 1 & b^2 & \frac{b}{ac} \\ 1 & c^2 & \frac{c}{ab} \end{vmatrix} = 0$

(iv) $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$

(v) $\begin{vmatrix} bc & ca & ab \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ a & b & c \end{vmatrix} = 0$

(vi) $\begin{vmatrix} mn & l & l^2 \\ nl & m & m^2 \\ lm & n & n^2 \end{vmatrix} = \begin{vmatrix} 1 & l^2 & l^3 \\ 1 & m^2 & m^3 \\ 1 & n^2 & n^3 \end{vmatrix}$

(vii) $\begin{vmatrix} 2a & 2b & 2c \\ a+b & 2b & b+c \\ a+c & b+c & 2c \end{vmatrix} = 0$

(vii) $\begin{vmatrix} 7 & 2 & 6 \\ 6 & 3 & 2 \\ -3 & 5 & 1 \end{vmatrix} = \begin{vmatrix} 7 & 2 & 7 \\ 6 & 3 & 5 \\ -3 & 5 & -3 \end{vmatrix} + \begin{vmatrix} 7 & 2 & -1 \\ 6 & 3 & -3 \\ -3 & 5 & 4 \end{vmatrix}$

(ix) $\begin{vmatrix} -a & 0 & c \\ 0 & a & -b \\ b & -c & 0 \end{vmatrix} = 0$

Solution:

$$\text{L.H.S.} = \begin{vmatrix} \alpha & \beta+\gamma & 1 \\ \beta & \gamma+\alpha & 1 \\ \gamma & \alpha+\beta & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \alpha+\beta+\gamma & \beta+\gamma & 1 \\ \beta+\gamma+\alpha & \gamma+\alpha & 1 \\ \gamma+\alpha+\beta & \alpha+\beta & 1 \end{vmatrix} \quad C_1 + C_2$$

Take $(\alpha+\beta+\gamma)$ common from C_1

$$= (\alpha+\beta+\gamma)\begin{vmatrix} 1 & \beta+\gamma & 1 \\ 1 & \gamma+\alpha & 1 \\ 1 & \alpha+\beta & 1 \end{vmatrix} \quad \because \ C_1 \text{ and } C_3 \text{ are same}$$

$$= (\alpha+\beta+\gamma) . 0$$

$$= 0$$

(ii) $$\begin{vmatrix} 1 & 2 & 3x \\ 2 & 3 & 6x \\ 3 & 5 & 9x \end{vmatrix} = 0$$

$$\text{L.H.S.} = \begin{vmatrix} 1 & 2 & 3x \\ 2 & 3 & 6x \\ 3 & 5 & 9x \end{vmatrix}$$

Take 3x common from C_3

$$= 3x \begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & 5 & 3 \end{vmatrix}$$

$$= 3x\,(0) = 0$$

As C_1 and C_3 is same so determinant will be zero.

(iii) $$\begin{vmatrix} 1 & a^2 & \frac{a}{bc} \\ 1 & b^2 & \frac{b}{ac} \\ 1 & c^2 & \frac{c}{ab} \end{vmatrix} = 0$$

$$\text{L.H.S.} = \begin{vmatrix} 1 & a^2 & \frac{a}{bc} \\ 1 & b^2 & \frac{b}{ac} \\ 1 & c^2 & \frac{c}{ab} \end{vmatrix}$$

multiplying C_3 by abc

$$= \frac{1}{abc} \begin{vmatrix} 1 & a^2 & \frac{a\,.\,abc}{bc} \\ 1 & b^2 & \frac{b\,.\,abc}{ac} \\ 1 & c^2 & \frac{c\,.\,abc}{ab} \end{vmatrix}$$

$$= \frac{1}{abc} \begin{vmatrix} 1 & a^2 & a^2 \\ 1 & b^2 & b^2 \\ 1 & c^2 & c^2 \end{vmatrix}$$

$$= 0$$

As C_2 and C_3 are same so determinant will be zero.

(iv) $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$

$$\text{L.H.S.} = \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$$

$$= \begin{vmatrix} a-b+b-c+c-a & b-c & c-a \\ b-c+c-a+a-b & c-a & a-b \\ c-a+a-b+b-c & a-b & b-c \end{vmatrix} \qquad C_1 + C_2 + C_3$$

$$= \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix} = 0 \qquad \because C_1 \text{ is zero.}$$

$$= 0$$

(v) $\begin{vmatrix} bc & ca & ab \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ a & b & c \end{vmatrix} = 0$

$$\text{L.H.S.} = \begin{vmatrix} bc & ca & ab \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ a & b & c \end{vmatrix}$$

multiplying R_2 by abc

$$= \frac{1}{abc}\begin{vmatrix} bc & ca & ab \\ \frac{abc}{a} & \frac{abc}{b} & \frac{abc}{c} \\ a & b & c \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} bc & ca & ab \\ bc & ca & ab \\ a & b & c \end{vmatrix}$$

$$= 0$$

As R_1 and R_2 are same so det will be zero.

(vi) $\begin{vmatrix} mn & l & l^2 \\ nl & m & m^2 \\ lm & n & n^2 \end{vmatrix} = \begin{vmatrix} 1 & l^2 & l^3 \\ 1 & m^2 & m^3 \\ 1 & n^2 & n^3 \end{vmatrix}$

$$\text{L.H.S.} = \begin{vmatrix} mn & l & l^2 \\ nl & m & m^2 \\ lm & n & n^2 \end{vmatrix}$$

multiplying R_1 by l R_2 by m, r_3 by n.

$$= \frac{1}{lmn}\begin{vmatrix} lmn & l^2 & l^3 \\ lmn & m^2 & m^3 \\ lmn & n^2 & n^3 \end{vmatrix}$$

Taking $l$mn common from C_1

$$= \frac{lmn}{lmn}\begin{vmatrix} 1 & l^2 & l^3 \\ 1 & m^2 & m^3 \\ 1 & n^2 & n^3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & l^2 & l^3 \\ 1 & m^2 & m^3 \\ 1 & n^2 & n^3 \end{vmatrix}$$

= R.H.S.

(vii) $$\begin{vmatrix} \mathbf{2a} & \mathbf{2b} & \mathbf{2c} \\ \mathbf{a+b} & \mathbf{2b} & \mathbf{b+c} \\ \mathbf{a+c} & \mathbf{b+c} & \mathbf{2c} \end{vmatrix} = \mathbf{0}$$

$$\text{L.H.S.} = \begin{vmatrix} 2a & 2b & 2c \\ a+b & 2b & b+c \\ a+c & b+c & 2c \end{vmatrix}$$

$$= \begin{vmatrix} 2a & 2b-2a & 2c-2a \\ a+b & 2b-(a+b) & b+c-(a+b) \\ a+c & b+c-(a+c) & 2c-(a+c) \end{vmatrix} \quad \begin{matrix} C_2 - C_1 \\ C_3 - C_1 \end{matrix}$$

$$= \begin{vmatrix} 2a & 2(b-a) & 2(c-a) \\ a+b & 2b-a-b & b+c-a-b \\ a+c & b+c-a-c & 2c-a-c \end{vmatrix}$$

$$= \begin{vmatrix} 2a & 2(b-a) & 2(c-a) \\ a+b & b-a & c-a \\ a+c & b-a & c-a \end{vmatrix}$$

Taking (b – a) common from C_2, and (c – a) from C_3.

$$= (b-a)(c-a)\begin{bmatrix} 2a & 2 & 2 \\ a+b & 1 & 1 \\ a+c & 1 & 1 \end{bmatrix}$$

$$= (b-a)(c-a)(0) \qquad C_2 \text{ and } C_3 \text{ is same.}$$

= 0 = R.H.S.

(viii) $$\begin{vmatrix} \mathbf{7} & \mathbf{2} & \mathbf{6} \\ \mathbf{6} & \mathbf{3} & \mathbf{2} \\ \mathbf{-3} & \mathbf{5} & \mathbf{1} \end{vmatrix} = \begin{vmatrix} \mathbf{7} & \mathbf{2} & \mathbf{7} \\ \mathbf{6} & \mathbf{3} & \mathbf{5} \\ \mathbf{-3} & \mathbf{5} & \mathbf{-3} \end{vmatrix} + \begin{vmatrix} \mathbf{7} & \mathbf{2} & \mathbf{-1} \\ \mathbf{6} & \mathbf{3} & \mathbf{-3} \\ \mathbf{-3} & \mathbf{5} & \mathbf{4} \end{vmatrix}$$

$$\text{L.H.S.} = \begin{vmatrix} 7 & 2 & 6 \\ 6 & 3 & 2 \\ -3 & 5 & 1 \end{vmatrix} = \begin{vmatrix} 7 & 2 & 7-1 \\ 6 & 3 & 5-3 \\ -3 & 5 & -3+4 \end{vmatrix}$$

by using property of determinant

$$= \begin{vmatrix} 7 & 2 & 7 \\ 6 & 3 & 5 \\ -3 & 5 & -3 \end{vmatrix} + \begin{vmatrix} 7 & 2 & -1 \\ 6 & 3 & -3 \\ -3 & 5 & 4 \end{vmatrix}$$

= R.H.S.

(ix) $\begin{vmatrix} -a & 0 & c \\ 0 & a & -b \\ b & -c & 0 \end{vmatrix} = 0$

$$\text{L.H.S} = \begin{vmatrix} -a & 0 & c \\ 0 & a & -b \\ b & -c & 0 \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} -ab & 0 & bc \\ 0 & ac & -bc \\ ab & -ac & 0 \end{vmatrix} \begin{matrix} bR_1 \\ cR_2 \\ aR_3 \end{matrix}$$

Taking ab common from C_1.
Taking ac common from C_2.
Taking bc common from C_3.

$$= \frac{1}{abc}(ab)(ac)(bc)\begin{vmatrix} -1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix}$$

$$= abc\begin{vmatrix} -1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix} \quad R_1 + R_2$$

$$= abc\begin{vmatrix} -1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix}$$

$$= abc\begin{vmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix} \quad R_1 + R_3$$

$= abc\,(0)$ $\quad\therefore$ all elements of R_1 are zero.
$= 0$
$=$ R.H.S

Q.6 Find value of x if

(i) $\begin{vmatrix} 3 & 1 & x \\ -1 & 3 & 4 \\ x & 1 & 0 \end{vmatrix} = -30$

(ii) $\begin{vmatrix} 1 & x-1 & 3 \\ -1 & x+1 & 2 \\ 2 & -2 & x \end{vmatrix} = 0$

(iii) $\begin{vmatrix} 1 & 2 & 1 \\ 2 & x & 2 \\ 3 & 6 & x \end{vmatrix} = 0$

Solution:

(i) $\begin{vmatrix} 3 & 1 & x \\ -1 & 3 & 4 \\ x & 1 & 0 \end{vmatrix} = -30$

Expanding this determinant by R_1

$\Rightarrow \quad 3\begin{vmatrix} 3 & 4 \\ 1 & 0 \end{vmatrix} - 1\begin{vmatrix} -1 & 4 \\ x & 0 \end{vmatrix} + x\begin{vmatrix} -1 & 3 \\ x & 1 \end{vmatrix} = -30$

$\Rightarrow \quad 3\big(3(0) - (4)(1)\big) - 1\big((-1)(0) - (x)(4)\big) + x\big((-1)(1) - (x)(3)\big) = -30$

$\Rightarrow \quad 3(-4) - 1(-4x) + x(-1 - 3x) = -30$

$\Rightarrow \quad -12 + 4x - x - 3x^2 = -30$

$\Rightarrow \quad -3x^2 + 3x - 12 = -30$

$\Rightarrow \quad -3(x^2 - x + 4) = -30$

$\Rightarrow \quad x^2 - x + 4 = \frac{-30}{-3} = 10$

$\Rightarrow \quad x^2 - x + 4 - 10 = 0$

$\Rightarrow \quad x^2 - x - 6 = 0$

$\Rightarrow \quad x^2 - 3x + 2x - 6 = 0$

$\Rightarrow \quad x(x-3) + 2(x-3) = 0$

$\Rightarrow \quad (x+2)(x-3) = 0$

$\Rightarrow \quad \boxed{x = -2, 3}$

(ii) $\begin{vmatrix} 1 & x-1 & 3 \\ -1 & x+1 & 2 \\ 2 & -2 & x \end{vmatrix} = 0$

Expanding this determinant by R_1

$1\begin{vmatrix} x+1 & 2 \\ -2 & x \end{vmatrix} - (x-1)\begin{vmatrix} -1 & 2 \\ 2 & x \end{vmatrix} + 3\begin{vmatrix} -1 & x+1 \\ 2 & -2 \end{vmatrix} = 0$

$\Rightarrow \quad 1\big(x(x+1) - (-2)(2)\big) - (x-1)\big((-1)(x) - (2)(2)\big) + 3\big((-1)(-2) - 2(x+1)\big) = 0$

$\Rightarrow \quad (x^2 + x + 4) - (x-1)(-x-4) + 3(2 - 2x - 2) = 0$

$\Rightarrow \quad x^2 + x + 4 - (-x^2 - 4x + x + 4) + 3(-2x) = 0$

$\Rightarrow \quad x^2 + x + 4 + x^2 + 4x - x - 4 - 6x = 0$

$\Rightarrow \quad 2x^2 - 2x = 0$

$\Rightarrow \quad 2x(x-1) = 0$

$\Rightarrow \quad 2x = 0 \qquad x - 1 = 0$

$\Rightarrow \quad \boxed{x = 0} \qquad \boxed{x = 1}$

(iii) $\begin{vmatrix} 1 & 2 & 1 \\ 2 & x & 2 \\ 3 & 6 & x \end{vmatrix} = 0$

Expanding this determinant by R_1

$$1\begin{vmatrix} x & 2 \\ 6 & x \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 3 & x \end{vmatrix} + 1\begin{vmatrix} 2 & x \\ 3 & 6 \end{vmatrix} = 0$$

$\Rightarrow \quad (x^2 - 12) - 2(2x - 6) + (12 - 3x) = 0$

$\Rightarrow \quad x^2 - 12 - 4x + 12 + 12 - 3x = 0$

$\Rightarrow \quad x^2 - 7x + 12 = 0$

$\Rightarrow \quad x^2 - 4x - 3x + 12 = 0$

$\Rightarrow \quad x(x - 4) - 3(x - 4) = 0$

$\Rightarrow \quad (x - 3)(x - 4) = 0$

$\Rightarrow \quad x - 3 = 0 \qquad x - 4 = 0$

$\Rightarrow \quad \boxed{x = 3} \qquad \boxed{x = 4}$

Q.7 Evaluate the following determinants:

(i) $\begin{vmatrix} 3 & 4 & 2 & 7 \\ 2 & 5 & 0 & 3 \\ 1 & 2 & -3 & 5 \\ 4 & 1 & -2 & 6 \end{vmatrix}$ **(ii)** $\begin{vmatrix} 2 & 3 & 1 & -1 \\ 4 & 0 & 2 & 1 \\ 5 & 2 & -1 & 6 \\ 3 & -7 & 2 & -2 \end{vmatrix}$

(iii) $\begin{vmatrix} -3 & 9 & 1 & 1 \\ 0 & 3 & -1 & 2 \\ 9 & 7 & -1 & 1 \\ -2 & 0 & 1 & -1 \end{vmatrix}$

Solution:

(i) $\begin{vmatrix} 3 & 4 & 2 & 7 \\ 2 & 5 & 0 & 3 \\ 1 & 2 & -3 & 5 \\ 4 & 1 & -2 & 6 \end{vmatrix}$

Interchange R_1 and R_3

$$= -\begin{vmatrix} 1 & 2 & -3 & 5 \\ 2 & 5 & 0 & 3 \\ 3 & 4 & 2 & 7 \\ 4 & 1 & -2 & 6 \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & 2 & -3 & 4 \\ 2-2(1) & 5-2(2) & 0-2(-3) & 3-2(5) \\ 3-3(1) & 4-3(2) & 2-3(-3) & 7-3(5) \\ 4-4(1) & 1-4(2) & -2-4(-3) & 6-4(5) \end{vmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \\ R_4 - 4R_1 \end{matrix}$$

$$= -\begin{vmatrix} 1 & 2 & -3 & 5 \\ 0 & 5-4 & 0+6 & 3-10 \\ 0 & 4-6 & 2+9 & 7-15 \\ 0 & 1-8 & -2+12 & 6-20 \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & 2 & -3 & 5 \\ 0 & 1 & 6 & -7 \\ 0 & -2 & 11 & -8 \\ 0 & -7 & 10 & -14 \end{vmatrix}$$

Expanding by C_1

$$= -\left[1\begin{vmatrix} 1 & 6 & -7 \\ -2 & 11 & -8 \\ -7 & 10 & -14 \end{vmatrix} - 0 + 0 - 0\right]$$

$$= -\begin{vmatrix} 1 & 6 & -7 \\ -2 & 11 & -8 \\ -7 & 10 & -14 \end{vmatrix}$$

Expanding by R_1

$$= -\left[1\begin{vmatrix} 11 & -8 \\ 10 & -14 \end{vmatrix} - 6\begin{vmatrix} -2 & -8 \\ -7 & -14 \end{vmatrix} + (-7)\begin{vmatrix} -2 & 11 \\ -7 & 10 \end{vmatrix}\right]$$

$$= -\left[\left((11)(-14) - (10)(-8)\right) - 6\left((-2)(-14) - (-7)(-8)\right) - 7\left[((-2)(10) - (11)(-7)\right)\right]$$

$$= -\left[-154 + 80 - 6(28 - 56) - 7(-20 + 77)\right]$$

$$= -\left[-74 - 6(-28) - 7(57)\right]$$

$$= 305$$

(ii) $$\begin{vmatrix} \mathbf{2} & \mathbf{3} & \mathbf{1} & \mathbf{-1} \\ \mathbf{4} & \mathbf{0} & \mathbf{2} & \mathbf{1} \\ \mathbf{5} & \mathbf{2} & \mathbf{-1} & \mathbf{6} \\ \mathbf{3} & \mathbf{-7} & \mathbf{2} & \mathbf{-2} \end{vmatrix}$$

Interchange C_1 and C_3

$$= -\begin{vmatrix} 1 & 3 & 2 & -1 \\ 2 & 0 & 4 & 1 \\ -1 & 2 & 5 & 6 \\ 2 & -7 & 3 & -2 \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & 3 & 2 & -1 \\ 2-2(1) & 0-2(3) & 4-2(2) & 1-2(-1) \\ -1+1 & 2+3 & 5+2 & 6-1 \\ 2-2(1) & -7-2(3) & 3-2(2) & -2-2(-1) \end{vmatrix} \quad \begin{matrix} R_2-2R_1 \\ R_3+R_1 \\ R_4-2R_1 \end{matrix}$$

$$= \begin{vmatrix} 1 & 3 & 2 & -1 \\ 0 & -6 & 0 & 3 \\ 0 & 5 & 7 & 5 \\ 0 & -13 & -1 & 0 \end{vmatrix}$$

Expanding by C_1

$$= -\left[1 \begin{vmatrix} -6 & 0 & 3 \\ 5 & 7 & 5 \\ -13 & -1 & 0 \end{vmatrix} - 0 + 0 - 0 \right]$$

$$= -\begin{vmatrix} -6 & 0 & 3 \\ 5 & 7 & 5 \\ -13 & -1 & 0 \end{vmatrix}$$

Expanding by R_1

$$= -\left[-6 \begin{vmatrix} 7 & 5 \\ -1 & 0 \end{vmatrix} - 0 + 3 \begin{vmatrix} 5 & 7 \\ -13 & -1 \end{vmatrix} \right]$$

$$= -[-6\left(0-(-1)(5)\right) + 3\left((5)(-1)-(-13)(7)\right)]$$

$$= -[-6(5) + 3(-5+91)]$$

$$= -[-30 + 3(86)] = -228$$

(iii) $\begin{vmatrix} \mathbf{-3} & \mathbf{9} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{3} & \mathbf{-1} & \mathbf{2} \\ \mathbf{9} & \mathbf{7} & \mathbf{-1} & \mathbf{1} \\ \mathbf{-2} & \mathbf{0} & \mathbf{1} & \mathbf{-1} \end{vmatrix}$

Interchange C_1 and C_3

$$= -\begin{vmatrix} 1 & 9 & -3 & 1 \\ -1 & 3 & 0 & 2 \\ -1 & 7 & 9 & 1 \\ 1 & 0 & -2 & -1 \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & 9 & -3 & 1 \\ -1+1 & 3+9 & 0-3 & 2+1 \\ -1+1 & 7+9 & 9-3 & 1+1 \\ 1-1 & 0-9 & -2+3 & -1-1 \end{vmatrix} \quad \begin{matrix} R_2-2R_1 \\ R_3-3R_1 \\ R_4-4R_1 \end{matrix}$$

$$= -\begin{vmatrix} 1 & 9 & -3 & 1 \\ 0 & 12 & -3 & 3 \\ 0 & 16 & 6 & 2 \\ 0 & -9 & 1 & -2 \end{vmatrix}$$

Expanding by C_1

$$= -1\begin{vmatrix} 12 & -3 & 3 \\ 16 & 6 & 2 \\ -9 & 1 & -2 \end{vmatrix}$$

Expanding by R_1

$$= -\left[12\begin{vmatrix} 6 & 2 \\ 1 & -2 \end{vmatrix} - (-3)\begin{vmatrix} 16 & 2 \\ -9 & -2 \end{vmatrix} + 3\begin{vmatrix} 16 & 6 \\ -9 & 1 \end{vmatrix}\right]$$

$$= -[12(-12-2) + 3(-32+18) + 3(16+54)]$$

$$= -[12(-14) + 3(-14) + 3(70)]$$

$$= -[-210 + 210] = 0$$

Q.8 Show that $\begin{vmatrix} x & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{vmatrix} = (x+3)(x-1)^3$

Solution:

L.H.S.

$$\text{L.H.S.} = \begin{vmatrix} x & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{vmatrix}$$

$$= \begin{vmatrix} x+1+1+1 & 1 & 1 & 1 \\ 1+x+1+1 & x & 1 & 1 \\ 1+1+x+1 & 1 & x & 1 \\ 1+1+1+x & 1 & 1 & x \end{vmatrix} \quad C_1 + C_2 + C_3 + C_4$$

$$= \begin{vmatrix} x+3 & 1 & 1 & 1 \\ x+3 & x & 1 & 1 \\ x+3 & 1 & x & 1 \\ x+3 & 1 & 1 & x \end{vmatrix}$$

Take $(x+3)$ common from C_1

$$= (x+3)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{vmatrix}$$

$$= (x+3)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & x-1 & 0 & 0 \\ 0 & 0 & x-1 & 0 \\ 0 & 0 & 0 & x-1 \end{vmatrix} \quad \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \\ R_4 - R_1 \end{matrix}$$

Expanding by C_1

$$= (x+3)\begin{bmatrix} x-1 & 0 & 0 \\ 0 & x-1 & 0 \\ 0 & 0 & x-1 \end{bmatrix}$$

Expanding by C_1

$$= (x+3)\left[(x-1)\begin{bmatrix} x-1 & 0 \\ 0 & x-1 \end{bmatrix}\right]$$

$$= (x+3)\,[(x-1)\,((x-1)(x-1)-(0)(0))]$$

$$= (x+3)\,[(x-1)(x-1)(x-1)]$$

$$= (x+3)(x-1)^3$$

$$= \text{R H.S.}$$

Q.9 Find $|AA^t|$ and $|A^tA|$ if

(i) If $A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 3 & 4 \\ 2 & 1 \\ 1 & 1 \\ 2 & 3 \end{bmatrix}$

Solution:

(i) $A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}$

$$A^t = \begin{bmatrix} 3 & 2 \\ 2 & 1 \\ -1 & 3 \end{bmatrix}$$

$$AA^t = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 2 & 1 \\ -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} (3)(3)+(2)(2)+(-1)(-1) & (3)(2)+(2)(1)+(-1)(3) \\ (2)(3)+(1)(2)+(3)(-1) & (2)(2)+(1)(1)+(3)(3) \end{bmatrix}$$

$$= \begin{bmatrix} 9+4+1 & 6+2-3 \\ 6+2-3 & 4+1+9 \end{bmatrix} = \begin{bmatrix} 14 & 5 \\ 5 & 14 \end{bmatrix}$$

Now $|AA^t| = \begin{bmatrix} 14 & 5 \\ 5 & 14 \end{bmatrix} = (14)(14) - 5(5) = 196 - 25 = 171$

Now $A^tA = \begin{bmatrix} 3 & 2 \\ 2 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}$

$$= \begin{bmatrix} (3)(3)+(2)(2) & (3)(2)+(2)(1) & (3)(-1)+(2)(3) \\ (2)(3)+(1)(2) & (2)(2)+(1)(1) & (2)(-1)+(1)(3) \\ (-1)(3)+(3)(2) & (-1)(2)+(3)(1) & (-1)(-1)+(3)(3) \end{bmatrix}$$

$$= \begin{bmatrix} 9+4 & 6+2 & -3+6 \\ 6+2 & 4+1 & -2+3 \\ -3+6 & -2+3 & 1+9 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & 8 & 3 \\ 8 & 5 & 1 \\ 3 & 1 & 10 \end{bmatrix}$$

$$|A^t A| = \begin{vmatrix} 13 & 8 & 3 \\ 8 & 5 & 1 \\ 3 & 1 & 10 \end{vmatrix}$$

Expanding by R_1

$$= 13 \begin{vmatrix} 5 & 1 \\ 1 & 10 \end{vmatrix} - 8 \begin{vmatrix} 8 & 1 \\ 3 & 10 \end{vmatrix} + 3 \begin{vmatrix} 8 & 5 \\ 3 & 1 \end{vmatrix}$$

$$= 13(50-1) - 8(80-3) + 3(8-15)$$

$$= 13(49) - 8(77) + 3(-7)$$

$$= 637 - 616 - 21$$

$$= 637 - 637 = 0$$

(ii) $\mathbf{A} = \begin{bmatrix} \mathbf{3} & \mathbf{4} \\ \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \\ \mathbf{2} & \mathbf{3} \end{bmatrix}$

$$A^t = \begin{bmatrix} 3 & 2 & 1 & 2 \\ 4 & 1 & 1 & 3 \end{bmatrix}$$

$$A A^t = \begin{bmatrix} 3 & 4 \\ 2 & 1 \\ 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 & 2 \\ 4 & 1 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 9+16 & 6+4 & 3+4 & 6+12 \\ 6+4 & 4+1 & 2+1 & 4+3 \\ 3+4 & 2+1 & 1+1 & 2+3 \\ 6+12 & 4+3 & 2+3 & 4+9 \end{bmatrix}$$

$$|A A^t| = \begin{vmatrix} 25 & 10 & 7 & 18 \\ 10 & 5 & 3 & 7 \\ 7 & 3 & 2 & 5 \\ 18 & 7 & 5 & 13 \end{vmatrix}$$

$$= \begin{vmatrix} 25-3(7) & 10-3(3) & 7-3(2) & 18-3(5) \\ 10 & 5 & 3 & 7 \\ 7 & 3 & 2 & 5 \\ 18-2(7) & 7-2(3) & 5-2(2) & 13-2(5) \end{vmatrix} \quad \begin{matrix} R_1 - 3R_3 \\ R_4 - 2R_3 \end{matrix}$$

$$= \begin{vmatrix} 4 & 1 & 1 & 3 \\ 10 & 5 & 3 & 7 \\ 7 & 3 & 2 & 5 \\ 4 & 1 & 1 & 3 \end{vmatrix}$$

As R_1 and R_4 are same so det will be zero.

$$A^t A = \begin{bmatrix} 3 & 2 & 1 & 2 \\ 4 & 1 & 1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 2 & 1 \\ 1 & 1 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} (3)(3)+(2)(2)+(1)(1)+(2)(2) & (3)(4)+(2)(1)+(1)(1)+(2)(3) \\ (4)(3)+(1)(2)+(1)(1)+(3)(2) & (4)(4)+(1)(1)+(1)(1)+(3)(3) \end{bmatrix}$$

$$= \begin{bmatrix} 9+4+1+4 & 12+2+1+6 \\ 12+2+1+6 & 16+1+1+9 \end{bmatrix}$$

$$= \begin{bmatrix} 18 & 21 \\ 21 & 27 \end{bmatrix}$$

$$|A^t A| = \begin{vmatrix} 18 & 21 \\ 21 & 27 \end{vmatrix}$$

$$= (18)(27) - (21)(21) = 486 - 441 = 45$$

Q.10 If 'A' is a square matrix of order 3, then show that $|KA| = K^3 |A|$

Solution:

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$k A = \begin{bmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{bmatrix}$$

$$|kA| = \begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{vmatrix}$$

Take 'k' common from R_1, R_2 and R_3

$$= k.k.k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$|k A| = k^3 |A| =$ R.H.S.

Hence proved.

Q.11 Find value of 'λ' if A and B are singular

(i) $A = \begin{bmatrix} 4 & \lambda & 3 \\ 7 & 3 & 6 \\ 2 & 3 & 1 \end{bmatrix}$

(ii) $B = \begin{bmatrix} 5 & 1 & 2 & 0 \\ 8 & 2 & 5 & 1 \\ 3 & 2 & 0 & 1 \\ 2 & \lambda & -1 & 3 \end{bmatrix}$

Solution:

(i) $A = \begin{bmatrix} 4 & \lambda & 3 \\ 7 & 3 & 6 \\ 2 & 3 & 1 \end{bmatrix}$

As A is singular so $|A| = 0$

$\Rightarrow \quad \begin{vmatrix} 4 & \lambda & 3 \\ 7 & 3 & 6 \\ 2 & 3 & 1 \end{vmatrix} = 0$

Expanding by R_1

$\Rightarrow \quad 4\begin{vmatrix} 3 & 6 \\ 3 & 1 \end{vmatrix} - \lambda\begin{vmatrix} 7 & 6 \\ 2 & 1 \end{vmatrix} + 3\begin{vmatrix} 7 & 3 \\ 2 & 3 \end{vmatrix} = 0$

$\Rightarrow \quad 4(3-18) - \lambda(7-12) + 3(21-6) = 0$

$\Rightarrow \quad 4(-15) - \lambda(-5) + 3(15) = 0$

$\Rightarrow \quad -60 + 5\lambda + 45 = 0$

$\Rightarrow \quad -15 + 5\lambda = 0 \Rightarrow 5(-3+\lambda) = 0$

$\Rightarrow \quad -3 + \lambda = 0 \Rightarrow \boxed{\lambda = 3}$

(ii) $B = \begin{bmatrix} 5 & 1 & 2 & 0 \\ 8 & 2 & 5 & 1 \\ 3 & 2 & 0 & 1 \\ 2 & \lambda & -1 & 3 \end{bmatrix}$

As 'B' is a singular so $|B| = 0$

$\Rightarrow \quad \begin{vmatrix} 5 & 1 & 2 & 0 \\ 8 & 2 & 5 & 1 \\ 3 & 2 & 0 & 1 \\ 2 & \lambda & -1 & 3 \end{vmatrix} = 0$

Interchange C_1 and C_4

$$\Rightarrow \quad -\begin{vmatrix} 0 & 1 & 2 & 5 \\ 1 & 2 & 5 & 8 \\ 1 & 2 & 0 & 3 \\ 3 & \lambda & -1 & 2 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} 0 & 1 & 2 & 5 \\ 1 & 2 & 5 & 8 \\ 1-1 & 2-2 & 0-5 & 3-8 \\ 3-3(1) & \lambda-3(2) & -1-3(5) & 2-3(8) \end{vmatrix} = 0 \qquad \begin{matrix} R_3 - R_2 \\ R_4 - 3R_2 \end{matrix}$$

$$\Rightarrow \quad -\begin{vmatrix} 0 & 1 & 2 & 5 \\ 1 & 2 & 5 & 8 \\ 0 & 0 & -5 & -5 \\ 0 & \lambda-6 & -16 & -22 \end{vmatrix} = 0$$

Expanding by C_1

$$\Rightarrow \quad (-1)(-1)\begin{vmatrix} 1 & 2 & 5 \\ 0 & -5 & -5 \\ \lambda-6 & -16 & -22 \end{vmatrix} = 0$$

Expanding by C_1

$$\Rightarrow \quad \begin{vmatrix} -5 & -5 \\ -16 & -22 \end{vmatrix} - 0 + (\lambda-6)\begin{vmatrix} 2 & 5 \\ -5 & -5 \end{vmatrix} = 0$$

$\Rightarrow \quad (110-80) + (\lambda-6)(15) = 0$

$\Rightarrow \quad 30 + (\lambda-6)(15) = 0$

$\Rightarrow \quad 30 + 15\lambda - 90 = 0$

$\Rightarrow \quad 15\lambda - 60 = 0$

$\Rightarrow \quad 15(\lambda-4) = 0$

$\Rightarrow \quad \lambda - 4 = 0 \Rightarrow \boxed{\lambda = 4}$

Q.12 Which of the following matrices are singular and which of them are non–singular.

(i) $\begin{vmatrix} 1 & 0 & 3 \\ 3 & 1 & -1 \\ 0 & 2 & 4 \end{vmatrix}$ **(ii)** $\begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{vmatrix}$

(iii) $\begin{vmatrix} 1 & 1 & 2 & -1 \\ 1 & 2 & -1 & -3 \\ 2 & 3 & 1 & 2 \\ 3 & -1 & 3 & 4 \end{vmatrix}$

Solution:

(i) $\mathbf{A} = \begin{vmatrix} 1 & 0 & 3 \\ 3 & 1 & -1 \\ 0 & 2 & 4 \end{vmatrix}$

Expanding by R_1

$= 1(4-(-2)) - 0 + 3(6-0)$

$= 4 + 2 + 3(6) = 6 + 18 = 24 \neq 0$

$\Rightarrow$ A is non–singular.

(ii) $\mathbf{A} = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{bmatrix}$

$|A| = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{bmatrix}$

Expanding by R_1

$= 2(5-0) - 3(5-0) + (-1)(-3-2)$

$= 2(5) - 3(5) - 1(-5)$

$= 10 - 15 + 5 = 0 \Rightarrow$ A is singular.

(iii) $\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 & -1 \\ 1 & 2 & -1 & -3 \\ 2 & 3 & 1 & 2 \\ 3 & -1 & 3 & 4 \end{bmatrix}$

Let

$|A| = \begin{vmatrix} 1 & 1 & 2 & -1 \\ 1 & 2 & -1 & -3 \\ 2 & 3 & 1 & 2 \\ 3 & -1 & 3 & 4 \end{vmatrix}$

$= \begin{vmatrix} 1 & 1 & 2 & -1 \\ 1-1 & 2-1 & -1-2 & -3-(-1) \\ 2-2(1) & 3-2(1) & 1-2(2) & 2-2(-1) \\ 3-3(1) & -1-3(1) & 3-3(2) & 4-3(-1) \end{vmatrix} \quad \begin{matrix} R_2 - R_1 \\ R_3 - 2R_1 \\ R_4 - 3R_1 \end{matrix}$

$= \begin{vmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -3 & -2 \\ 0 & 1 & -3 & 4 \\ 0 & -4 & -3 & 7 \end{vmatrix}$

Expanding by C_1

$$= 1 . \begin{vmatrix} 1 & -3 & -2 \\ 1 & -3 & 4 \\ -4 & -3 & 7 \end{vmatrix}$$

Expanding by R_1

$$= 1 \begin{vmatrix} -3 & 4 \\ -3 & 7 \end{vmatrix} - (-3) \begin{vmatrix} 1 & 4 \\ -4 & 7 \end{vmatrix} -2 \begin{vmatrix} 1 & -3 \\ -4 & -3 \end{vmatrix}$$

$= \left(-21 - (-12)\right) + 3\left(7 - (-16)\right) - 2(-3 - 12)$

$= -21 + 12 + 3(7 + 16) - 2(-15)$

$= -21 + 12 + 69 + 30$

$= 90 \neq 0$

$\Rightarrow$ A is non–singular.-

Q.13 Find inverse of $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 3 \\ 2 & -4 & 1 \end{bmatrix}$ **and show that** $A^{-1} A = I_3$.

Solution:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{bmatrix}$$

As $A^{-1} = \dfrac{\text{adj}}{|A|}$ ……… (1)

$$|A| = 2 \begin{vmatrix} 1 & 0 \\ -3 & 5 \end{vmatrix} - 1 \begin{vmatrix} 1 & 0 \\ 2 & 5 \end{vmatrix} + 0$$

$= 2(5 - 0) - 1(5 - 0)$

$= 10 - 5 = 5 \neq 0$

Now

$$\text{Adj } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^t \quad \text{……… (2)}$$

where

$$A_{11} = (-1)^{1+1} M_{11} = (-1)^2 \begin{vmatrix} 1 & 0 \\ -3 & 5 \end{vmatrix}$$

$= 5 - 0 = 5$

$$A_{12} = (-1)^{1+2}\ M_{12} = (-1)^3 \begin{vmatrix} 1 & 0 \\ -3 & 5 \end{vmatrix}$$
$$= -(5-0) = -5$$
$$A_{13} = (-1)^{1+3}\ M_{13} = (-1)^4 \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix}$$
$$= (-3)-2 = -3-2 = -5$$
$$A_{21} = (-1)^{2+1}\ M_{21} = (-1)^3 \begin{vmatrix} 1 & 0 \\ -3 & 5 \end{vmatrix}$$
$$= -(5-0) = -5$$
$$A_{22} = (-1)^{2+2}\ M_{22} = (-1)^4 \begin{vmatrix} 2 & 0 \\ 2 & 5 \end{vmatrix}$$
$$=10-0 = 10$$
$$A_{23} = (-1)^{2+3}\ M_{23} = (-1)^5 \begin{vmatrix} 2 & 1 \\ 2 & -3 \end{vmatrix}$$
$$= -(-6-2) = -(-8) = 8$$
$$A_{31} = (-1)^{3+1}\ M_{31} = (-1)^4 \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}$$
$$= 0$$
$$A_{32} = (-1)^{3+2}\ M_{32} = (-1)^5 \begin{vmatrix} 2 & 0 \\ 1 & 0 \end{vmatrix}$$
$$= (-1)(0) = 0$$
$$A_{33} = (-1)^{3+3}\ M_{33} = (-1)^6 \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$$
$$= (+1)(2-1) = 1$$

Put values in (2)

$$\text{Adj } A = \begin{bmatrix} 5 & -5 & -5 \\ -5 & 10 & 8 \\ 0 & 0 & 1 \end{bmatrix}^t = \begin{bmatrix} 5 & -5 & 0 \\ -5 & 10 & 0 \\ -5 & 8 & 1 \end{bmatrix}$$

Put in (1)

$$A^{-1} = \frac{1}{5}\begin{bmatrix} 5 & -5 & 0 \\ -5 & 10 & 0 \\ -5 & 8 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{5}{5} & -\frac{5}{5} & \frac{0}{5} \\ -\frac{5}{5} & \frac{10}{5} & \frac{0}{5} \\ -\frac{5}{5} & \frac{8}{5} & \frac{1}{5} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ -1 & \frac{8}{5} & \frac{1}{5} \end{bmatrix}$$

To show $A^{-1}A = I_3$

$$A^{-1}A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ -1 & \frac{8}{5} & \frac{1}{5} \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} (1)(2) + (-1)(1) + (0)(2) & (1)(1) + (-1)(1) + (0)(-3) & (1)(0) + (-1)(0) + (0)(5) \\ (-1)(2) + (2)(1) + (0)(2) & (-1)(1) + (2)(1) + (0)(-3) & (-1)(0) + (2)(0) + (0)(5) \\ (-1)(2) + \left(\frac{8}{5}\right)(1) + \left(\frac{1}{5}\right)(2) & (-1)(1) + \left(\frac{8}{5}\right)(1) + \left(\frac{1}{5}\right)(-3) & (-1)(0) + \left(\frac{8}{5}\right)(0) + \left(\frac{1}{5}\right)(5) \end{bmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = I_3$$

Q.14 Verify that $(AB)^{-1} = B^{-1} . A^{-1}$ if

(i) $A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} -3 & 1 \\ 4 & -1 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 5 & 1 \\ 2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$

Solution:

$$AB = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -3 & 1 \\ 4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} (1)(-3) + (2)(4) & (1)(1) + (2)(-1) \\ (-1)(-3) + (0)(4) & (-1)(1) + (0)(-1) \end{bmatrix}$$

$$= \begin{bmatrix} -3+8 & 1-2 \\ 3+0 & -1-0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 \\ 3 & -1 \end{bmatrix}$$

As

$$(AB)^{-1} = \frac{\text{adj }(AB)}{\det (AB)} \quad \text{......... (1)}$$

$$\det (AB) = |AB| = \begin{vmatrix} -1 & 1 \\ -3 & -5 \end{vmatrix} = -5 - (3) = -5 + 3 = -2$$

$$\text{adj }(AB) = \begin{bmatrix} -1 & 1 \\ -3 & 5 \end{bmatrix}$$

Put values in equation (1)

$$(AB)^{-1} = -\frac{1}{2}\begin{bmatrix} -1 & 1 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & -\frac{5}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & -\frac{5}{2} \end{bmatrix}$$

Now

$$B^{-1} = \frac{\text{adj}B}{|B|} \qquad \ldots\ldots\ldots (2)$$

$$|B| = \begin{vmatrix} -3 & 1 \\ 4 & -1 \end{vmatrix} = (-3)(-1) - (4)(1) = 3 - 4 = -1$$

$$\text{AdJ } B = \begin{bmatrix} -1 & -1 \\ -4 & -3 \end{bmatrix}$$

Put values in equation (2)

$$B^{-1} = \frac{1}{-1}\begin{bmatrix} -1 & -1 \\ -4 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 3 \end{bmatrix}$$

Now

$$A^{-1} = \frac{\text{adj}A}{|A|} \qquad \ldots\ldots\ldots (3)$$

$$|A| = \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} = (0)(1) - (2)(-1) = 2$$

$$\text{adJ} = \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix}$$

put values in equation (3)

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{0}{2} & \frac{-2}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 0 & -1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$B^{-1}A^{-1} = \begin{bmatrix} 1 & 1 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} (1)(0)+(1)\left(\frac{1}{2}\right) & (1)(-1)+(1)\left(\frac{1}{2}\right) \\ (4)(0)+(3)\left(\frac{1}{2}\right) & (4)(-1)+(3)\left(\frac{1}{2}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & -1+\frac{1}{2} \\ \frac{3}{2} & -4+\frac{3}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & -\frac{5}{2} \end{bmatrix} \quad \text{......... (II)}$$

From (I) & (II)

$(AB)^{-1} = B^{-1}A^{-1}$ proved.

Q.15 Verify that $(AB)^t = B^t A^t$ and if

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 1 \\ 3 & 2 \\ 0 & -1 \end{bmatrix}$$

Solution:

Given that

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 3 & 2 \\ 0 & -1 \end{bmatrix}$$

$$(AB) = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 3 & 2 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} (1)(1)+(-1)(3)+(2)(0) & (1)(1)+(-1)(2)+(2)(-1) \\ (0)(1)+(3)(3)+(1)(0) & (0)(1)+(3)(2)+(1)(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 1-3 & 1-2-2 \\ 9+0 & 6-1 \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ 9 & 5 \end{bmatrix}$$

$$(AB)^t = \begin{bmatrix} -2 & 9 \\ -3 & 5 \end{bmatrix} \quad \text{......... (a)}$$

Now

$$B^t = \begin{bmatrix} 1 & 3 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

$$A^t = \begin{bmatrix} 1 & 0 \\ -1 & 3 \\ 2 & 1 \end{bmatrix}$$

$$B^t . A^t = \begin{bmatrix} 1 & 3 & 0 \\ 1 & 2 & -1 \end{bmatrix} \begin{vmatrix} 1 & 0 \\ -1 & 3 \\ 2 & 1 \end{vmatrix}$$

$$= \begin{bmatrix} (1)(1) + (3)(-1) + (0)(2) & (1)(0) + (3)(3) + (0)(1) \\ (1)(1) + (2)(-1) + (-1)(2) & (1)(0) + (2)(3) + (-1)(1) \end{bmatrix}$$

$$= \begin{bmatrix} 1-3 & 9 \\ 1-2-2 & 6-1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 9 \\ -3 & 5 \end{bmatrix} \quad \text{......... (b)}$$

From (a) & (b)

$(AB)^t = B^t A^t$

Q.16 Verify that $(A^{-1})^t = (A^t)^{-1}$ if $A = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$

Solution:

Given that

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adJ } A}{|A|} \quad \text{......... (1)}$$

$$|A| = \begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} = 2 - (-3) = 5$$

$$\text{AdJ } A = \begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix}$$

Put values in (1)

$$A^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ -\frac{3}{5} & \frac{2}{5} \end{bmatrix}$$

$$(A^{-1})^t = \begin{bmatrix} \frac{1}{5} & -\frac{3}{5} \\ \frac{1}{5} & \frac{2}{5} \end{bmatrix} \quad \text{......... (a)}$$

Now

$$A^t = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$$

$$(A^t)^{-1} = \frac{\text{adJ } (A^t)}{|A^t|} \quad \text{......... (2)}$$

$$\text{adj } A^t = \begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix}$$

$$|A^t| = \begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} = 2-(-3) = 5$$

$$\text{Adj } A^t = \begin{vmatrix} 1 & -3 \\ 1 & 2 \end{vmatrix}$$

Put values in (2)

$$(A^t)^{-1} = \frac{1}{5}\begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{vmatrix} \frac{1}{5} & -\frac{3}{5} \\ \frac{1}{5} & \frac{2}{5} \end{vmatrix} \quad \text{......... (B)}$$

From (a) & (b)

$(A^{-1})^t = (A^t)^{-1}$

Hence proved.

Q.17 If A and B are non–singular matrices, then show that

(i) $(AB)^{-1} = B^{-1}A^{-1}$ (ii) $(A^{-1})^{-1} = A$

Solution:

(i) $(AB)^{-1} = B^{-1}A^{-1}$

We know that.

$AA^{-1} = I$

$BB^{-1} = I$

$(AB)(AB)^{-1} = I$ .... (1)

We take

$(AB)(B^{-1}A^{-1}) = I$ .... (2)

L.H.S $= (AB)(B^{-1}A^{-1})$

$= A(BB^{-1})A^{-1}$

$= AIA^{-1}$

$= AA^{-1}$

$= I$

$=$ R.H.S

From equations (1) and (2), we have $(AB)^{-1} = B^{-1}A^{-1}$

$\Rightarrow$

$B^{-1}A^{-1}$ is inverse of AB.

$\Rightarrow$

$(AB)^{-1} = B^{-1}A^{-1}$

(ii) $(A^{-1})^{-1} = A$

As $AA^{-1} = I$ and $A^{-1}A = I$

$\Rightarrow$

A^{-1} is inverse of A

so $(A^{-1})^{-1} = A.$ Hence proved.