Calculus Appendix

Implicit Differentiation, p. 564

1. The chain rule states that if y is a composite function, then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$. To differentiate an equation implicitly, first differentiate both sides of the equation with respect to x, using the chain rule for terms involving y, then solve for $\frac{dy}{dx}$.

2. a. $\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(36)$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(36)$$
$$2x + \frac{dy^2}{dy} \times \frac{dy}{dx} = 0$$
$$2x + 2y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = -\frac{x}{y}$$

b. $\frac{d}{dx}(15y^2) = \frac{d}{dx}(2x^3)$

$$15\frac{dy^2}{dy} \times \frac{dy}{dx} = \frac{d}{dx}(2x^3)$$
$$30y\frac{dy}{dx} = 6x^2$$
$$\frac{dy}{dx} = \frac{x^2}{5y}$$

c. $3xy^2 + y^3 = 8$

$$3y^2 + 3x(2y)\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$$
$$\frac{dy}{dx}(2xy + y^2) = -y^2$$
$$\frac{dy}{dx} = \frac{-y^2}{2xy + y^2}$$

d. $9x^2 - 16y^2 = -144$

$$18x - 32y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{9x}{16y}$$

e. $\frac{x^2}{16} + \frac{3}{13}y^2 = 1$

$$\frac{2x}{16} + \frac{6}{13}y\frac{dy}{dx} = 0$$
$$26x + 96y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = -\frac{13x}{48y}$$

f. $\frac{d}{dx}(x^2 + y^2 + 5y) = \frac{d}{dx}(10)$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) + \frac{d}{dx}(5y) = \frac{d}{dx}(10)$$
$$2x + \frac{dy^2}{dy} \times \frac{dy}{dx} + 5\frac{dy}{dy} \times \frac{dy}{dx} = 0$$
$$2x + \frac{dy}{dx}(2y + 5) = 0$$
$$\frac{dy}{dx} = -\frac{2x}{2y + 5}$$

3. a. $x^2 + y^2 = 13$

$$2x + 2y\frac{dy}{dx} = 0$$

At $(2, -3)$,

$$2(2) + 2(-3)\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{2}{3}.$$

The equation of the tangent at $(2, -3)$ is $y = \frac{2}{3}x + b$.

At $(2, -3)$,

$$-3 = \frac{2}{3}(2) + b$$
$$-9 = 4 + 3b$$
$$-13 = 3b$$
$$-\frac{13}{3} = b.$$

Therefore, the equation of the tangent to $x^2 + y^2 = 13$ is $y = \frac{2}{3}x - \frac{13}{3}$.

b. $\frac{d}{dx}(x^2 + 4y^2) = \frac{d}{dx}(100)$

$$\frac{d}{dx}(x^2) + 4\frac{d}{dx}(y^2) = \frac{d}{dx}(100)$$
$$2x + 4\frac{dy^2}{dy} \times \frac{dy}{dx} = 0$$

$$2x + 8y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = -\frac{x}{4y}$$

At the point $(-8, 3)$, the slope of the tangent is therefore $\frac{dy}{dx} = -\frac{-8}{4(3)} = \frac{2}{3}$. The equation of the tangent line is therefore

$$y = \frac{2}{3}(x + 8) + 3$$

c. $\frac{x^2}{25} - \frac{y^2}{36} = -1$

$$\frac{2x}{25} - \frac{2y}{36}\frac{dy}{dx} = 0$$
$$36x - 25y\frac{dy}{dx} = 0$$

At $(5\sqrt{3}, -12)$,

$$180\sqrt{3} + 300\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = -\frac{3\sqrt{3}}{5}.$$

The equation of the tangent is $y = mx + b$.

At $(5\sqrt{3}, -12)$ and with $m = -\frac{3\sqrt{3}}{5}$,

$$-12 = -\frac{3\sqrt{3}}{5}(5\sqrt{3}) + b$$
$$-12 = -9 + b$$
$$-3 = b$$

Therefore, the equation of the tangent is

$$y = -\frac{3\sqrt{3}}{5}x - 3.$$

d. $\frac{d}{dx}\left(\frac{x^2}{81} - \frac{5y^2}{162}\right) = \frac{d}{dx}(1)$

$$\left(\frac{1}{81}\right)\frac{d}{dx}(x^2) - \left(\frac{5}{162}\right)\frac{d}{dx}(y^2) = \frac{d}{dx}(1)$$
$$\left(\frac{1}{81}\right)2x - \left(\frac{5}{162}\right)\frac{dy^2}{dy} \times \frac{dy}{dx} = 0$$
$$\frac{2}{81}x - \frac{5}{81}y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{2x}{5y}$$

At the point $(-11, -4)$, the slope of the tangent is therefore $\frac{dy}{dx} = \frac{2(-11)}{5(-4)} = \frac{11}{10}$. The equation of the tangent line is therefore

$$y = \frac{11}{10}(x + 11) - 4$$

4. $x + y^2 = 1$

The line $x + 2y = 0$ has slope of $-\frac{1}{2}$.

$$1 + 2y\frac{dy}{dx} = 0$$

Since the tangent line is parallel to $x + 2y = 0$, then $\frac{dy}{dx} = -\frac{1}{2}$.

$$1 + 2y\left(-\frac{1}{2}\right) = 0$$
$$1 - y = 0$$
$$y = 1$$

Substituting,

$$x + 1 = 1$$
$$x = 0$$

Therefore, the tangent line to the curve $x + y^2 = 1$ is parallel to the line $x + 2y = 0$ at (0, 1).

5. a. $5x^2 - 6xy + 5y^2 = 16$

$$10x - \left(6y + \frac{dy}{dx}(6x)\right) + 10y\frac{dy}{dx} = 0(1)$$

At $(1, -1)$,

$$10 - \left(-6 + 6\frac{dy}{dx}\right) - 10\frac{dy}{dx} = 0$$
$$16 - 16\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = 1.$$

b. When the tangent line is horizontal, $\frac{dy}{dx} = 0$.

Substituting,

$10x - (6y + 0) + 0 = 0$.

$y = \frac{5}{3}x$ at the point (x_1, y_1) of tangency:

substitute $y_1 = \frac{5}{3}x_1$ into $5x_1^2 - 6x_1y_1 + 5y_1^2 = 16$.

$$5x_1^2 - 6x_1\left(\frac{5}{3}x_1\right) + 5\left(\frac{25}{9}x_1^2\right) = 16$$
$$45x_1^2 - 90x_1^2 + 125x_1^2 = 144$$
$$80x_1^2 = 144$$
$$5x_1^2 = 9$$

$$x_1 = \frac{3}{\sqrt{5}} \text{ or } x_1 = -\frac{3}{\sqrt{5}}$$
$$y_1 = \frac{5}{\sqrt{5}} \text{ or } y_1 = \sqrt{5}$$
$$y_1 = -\frac{5}{\sqrt{5}} \text{ or } y_1 = -\sqrt{5}$$

Therefore, the required points are $\left(\frac{3}{\sqrt{5}}, \sqrt{5}\right)$ and $\left(-\frac{3}{\sqrt{5}}, -\sqrt{5}\right)$.

6. $5x^2 + y^2 = 21$

$$10x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{5x}{y}$$

At $(-2, -1)$, $\frac{dy}{dx} = -\frac{-10}{-1}$ or -10.

The slope of the tangent line to the ellipse at $(-2, -1)$ is -10.

7. $x^3 + y^3 - 3xy = 17$

$$3x^2 + 3y^2\frac{dy}{dx} - \left[3y + \frac{dy}{dx}(3x)\right] = 0$$

At (2, 3),

$$12 + 27\frac{dy}{dx} - 9 - 6\frac{dy}{dx} = 0$$

$$21\frac{dy}{dx} = -3.$$

The slope of the tangent is $\frac{dy}{dx} = -\frac{1}{7}$.

Therefore, the slope of the normal at (2, 3) is 7.

The equation of the normal at (2, 3) is $\frac{y-3}{x-2} = 7$

$y - 3 = 7x - 14$ or $7x - y - 11 = 0$

8. $\frac{d}{dx}(y^2) = \frac{d}{dx}\left(\frac{x^3}{2-x}\right)$

$$\frac{dy^2}{dy} \times \frac{dy}{dx} = \frac{(2-x)(3x^2) - x^3(-1)}{(2-x)^2}$$

$$2y\frac{dy}{dx} = \frac{6x^2 - 2x^3}{(2-x)^2}$$

$$\frac{dy}{dx} = \frac{3x^2 - x^3}{y(2-x)^2}$$

At $(1, -1)$, $\frac{dy}{dx} = \frac{3(1)^2 - (1)^3}{(-1)(2-1)^2} = -2$

The slope of the normal is $\frac{1}{2}$. The equation of the normal at $(1, -1)$ is $y - (-1) = \frac{1}{2}(x - 1)$ or $y = \frac{1}{2}x - \frac{3}{2}$.

9. a. $\frac{d}{dx}(x+y)^3 = \frac{d}{dx}(12x)$

$$3(x+y)^2 \times \frac{d}{dx}(x+y) = 12$$

$$3(x+y)^2\left(1 + \frac{dy}{dx}\right) = 12$$

$$\frac{dy}{dx} = \frac{4}{(x+y)^2} - 1$$

b. $\frac{d}{dx}(\sqrt{x+y} - 2x) = \frac{d}{dx}(1)$

$$\frac{d}{dx}((x+y)^{\frac{1}{2}}) - \frac{d}{dx}(2x) = \frac{d}{dx}(1)$$

$$\frac{1}{2}(x+y)^{-\frac{1}{2}} \times \frac{d}{dx}(x+y) - 2 = 0$$

$$\frac{1}{2}(x+y)^{-\frac{1}{2}}\left(1 + \frac{dy}{dx}\right) = 2$$

$$\frac{dy}{dx} = 4\sqrt{x+y} - 1$$

10. $4x^2y - 3y = x^3$

a. $8xy + \frac{dy}{dx}(4x^2) - 3\frac{dy}{dx} = 3x^2$

$$\frac{dy}{dx}(4x^2 - 3) = 3x^2 - 8xy$$

$$\frac{dy}{dx} = \frac{3x^2 - 8xy}{4x^2 - 3} \qquad (1)$$

b. $y(4x^2 - 3) = x^3$

$$y = \frac{x^3}{4x^2 - 3}$$

$$\frac{dy}{dx} = \frac{3x^2(4x^2 - 3) - 8x(x^3)}{(4x^2-3)^2}$$

$$= \frac{12x^4 - 9x^2 - 8x^4}{(4x^2-3)^2}$$

$$= \frac{4x^4 - 9x^2}{(4x^2-3)^2} \qquad (2)$$

c. We must show that (1) is equivalent to (2).

From (1): $\frac{dy}{dx} = \frac{3x^2 - 8xy}{4x^2 - 3}$ and substituting,

$y = \frac{x^3}{4x^2 - 3}$

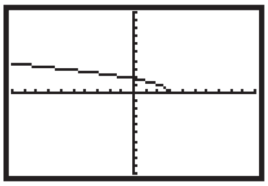
$$\frac{dy}{dx} = \frac{3x^2 - 8x\left(\frac{x^3}{4x^2-3}\right)}{4x^2 - 3}$$

$$= \frac{3x^2 - (4x^2 - 3) - 8x^4}{(4x^2-3)^2}$$

$$= \frac{12x^4 - 9x^2 - 8x^4}{(4x^2-3)^2}$$

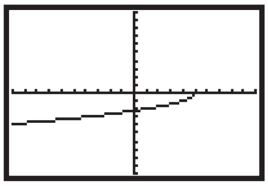
$$= \frac{4x^4 - 9x^2}{(4x^2-3)^2} = (2), \text{ as required.}$$

11. a.



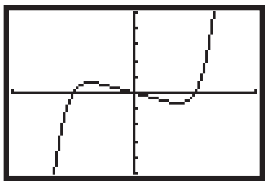
There is one tangent when $x = 1$.

b.



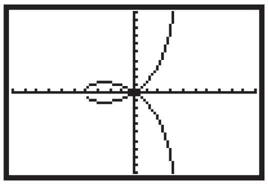
There is one tangent when $x = 1$.

c.



There is one tangent when $x = 1$.

d.



There are two tangents when $x = 1$.

12. $\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} = 10,\ x \neq y \neq 0,\ \frac{dy}{dx} = \frac{y}{x}$

$$\frac{1}{2}\left(\frac{x}{y}\right)^{-\frac{1}{2}}\frac{1y - \frac{dy}{dx}x}{y^2} + \frac{1}{2}\left(\frac{y}{x}\right)^{-\frac{1}{2}}\frac{\frac{dy}{dx}x - y}{x^2} = 0$$

$$\frac{y^{\frac{1}{2}}}{2x^{\frac{1}{2}}}\frac{1y - \frac{dy}{dx}x}{y^2} + \frac{x^{\frac{1}{2}}}{2y^{\frac{1}{2}}}\frac{\frac{dy}{dx}x - y}{x^2} = 0$$

Multiply by $2x^2y^2$:

$$x^{\frac{3}{2}}y^{\frac{1}{2}}\left(y - x\frac{dy}{dx}\right) + x^{\frac{1}{2}}y^{\frac{3}{2}}\left(\frac{dy}{dx}x - y\right) = 0$$

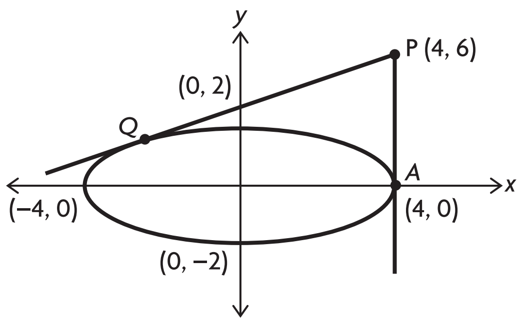
$$x^{\frac{3}{2}}y^{\frac{3}{2}} - x^{\frac{5}{2}}y^{\frac{1}{2}}\frac{dy}{dx} + x^{\frac{3}{2}}y^{\frac{3}{2}}\frac{dy}{dx} - x^{\frac{1}{2}}y^{\frac{5}{2}} = 0$$

$$\frac{dy}{dx}\left(x^{\frac{3}{2}}y^{\frac{3}{2}} - x^{\frac{3}{2}}y^{\frac{1}{2}}\right) = x^{\frac{1}{2}}y^{\frac{3}{2}} - x^{\frac{3}{2}}y^{\frac{3}{2}}$$

$$\frac{dy}{dx} = \frac{x^{\frac{1}{2}}y^{\frac{3}{2}}(y - x)}{x^{\frac{1}{2}}y^{\frac{1}{2}}(y - x)}$$

$$\frac{dy}{dx} = \frac{y}{x}, \text{ as required.}$$

13.



Let Q have coordinates

$(q, f(q)) = \left(q, \frac{\sqrt{16 - q^2}}{2}\right),\ q < 0.$

For $x^2 + 4y^2 = 16$

$$2x + 3y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{4y}.$$

At Q, $\frac{dy}{dx} = -\frac{q}{2\sqrt{16 - q^2}}$

The line through P has equation $\frac{y - 6}{x - 4} = m$.

Since PQ is the slope of the tangent line to $x^2 + 4y^2 = 16$, we conclude:

$m = \frac{dy}{dx}$ at point Q.

$$\frac{\frac{\sqrt{16 - q^2}}{2} - 6}{q - 4} = -\frac{q}{2\sqrt{16 - q^2}}$$

$$\frac{\sqrt{16 - q^2} - 12}{2(q - 4)} = -\frac{9}{2\sqrt{16 - q^2}}$$

$$16 - q^2 - 12\sqrt{16 - q^2} = -q(q - 4)$$

$$16 - q^2 - 12\sqrt{16 - q^2} = -q^2 + 4q$$

$$4 - q = 3\sqrt{16 - q^2}$$

$$16 - 8q + q^2 = 9(16 - q^2)$$

$$16 - 8q + q^2 = 144 - 9q^2$$

$$10q^2 - 8q - 128 = 0$$

$$5q^2 - 4q - 64 = 0$$

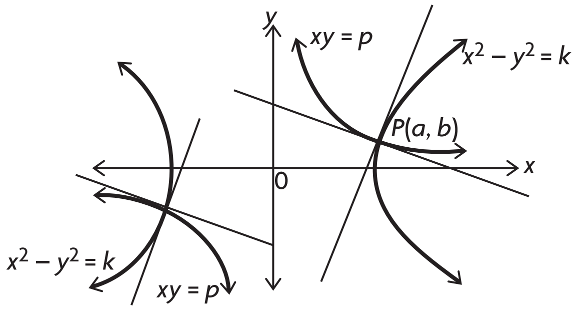
$$(5q + 16)(q - 4) = 0$$

$q = -\frac{16}{5}$ or $q = 4$ (as expected; see graph)

$f(q) = \frac{6}{5}$ or $f(q) = 0$

$$\frac{dy}{dx} = \frac{\frac{16}{5}}{4\left(\frac{6}{5}\right)} \text{ or } f(q) = 0$$
$$= \frac{2}{3}$$
Equation of the tangent at Q is $\frac{y-6}{x-4} = \frac{2}{3}$ or $2x - 3y + 10 = 0$ or equation of tangent at A is $x = 4$.

14.



Let $P(a, b)$ be the point of intersection where $a \neq 0$ and $b \neq 0$.

For $x^2 - y^2 = k$
$$2x - 2y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{x}{y}$$
At $P(a, b)$,
$$\frac{dy}{dx} = \frac{a}{b}.$$
For $xy = P$,
$$1 \cdot y + \frac{dy}{dx}x = P$$
$$\frac{dy}{dx} = -\frac{y}{x}$$
At $P(a, b)$,
$$\frac{dy}{dx} = -\frac{b}{a}.$$
At point $P(a, b)$, the slope of the tangent line of $xy = P$ is the negative reciprocal of the slope of the tangent line of $x^2 - y^2 = k$. Therefore, the tangent lines intersect at right angles, and thus, the two curves intersect orthogonally for all values of the constants k and P.

15. $\sqrt{x} + \sqrt{y} = \sqrt{k}$

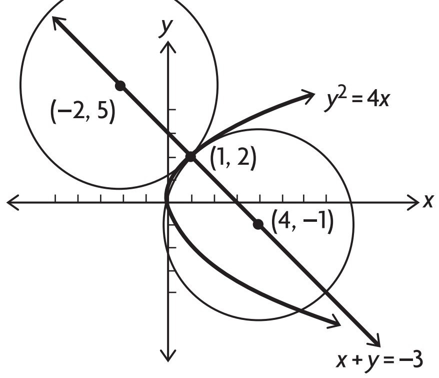
Differentiate with respect to x:
$$\frac{1}{2}x^{\frac{1}{2}} + \frac{1}{2}y^{\frac{1}{2}}\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$$
Let $P(a, b)$ be the point of tangency.
$$\frac{dy}{dx} - \frac{\sqrt{b}}{\sqrt{a}}$$
Equation of tangent line l at P is
$$\frac{y-b}{x-a} = -\frac{\sqrt{b}}{\sqrt{a}}.$$
x-intercept is found when $y = 0$.
$$\frac{-b}{x-a} = -\frac{\sqrt{b}}{\sqrt{a}}$$
$$-b\sqrt{a} = -\sqrt{b}x + a\sqrt{b}$$
$$x = \frac{a\sqrt{b} + b\sqrt{a}}{\sqrt{b}}$$
Therefore, the x-intercept is $\frac{a\sqrt{b} + b\sqrt{a}}{\sqrt{b}}$.

For the y-intercept, let $x = 0$,
$$\frac{y-b}{-a} = -\frac{\sqrt{b}}{\sqrt{a}}.$$
y-intercept is $\frac{a\sqrt{b}}{\sqrt{a}} + b$.

The sum of the intercepts is
$$\frac{a\sqrt{b} + b\sqrt{a}}{\sqrt{b}} + \frac{a\sqrt{b} + b\sqrt{a}}{\sqrt{a}}$$
$$= \frac{a^{\frac{3}{2}}b^{\frac{1}{2}} + 2ab + b^{\frac{3}{2}}a^{\frac{1}{2}}}{a^{\frac{1}{2}}b^{\frac{1}{2}}}$$
$$= \frac{a^{\frac{1}{2}}b^{\frac{1}{2}}(a + 2\sqrt{a}\sqrt{b} + b)}{a^{\frac{1}{2}}b^{\frac{1}{2}}}$$
$$= a + 2\sqrt{a}\sqrt{b} + b$$
$$= (a^{\frac{1}{2}} + b^{\frac{1}{2}})^2$$
Since $P(a, b)$ is on the curve, then $\sqrt{a} + \sqrt{b} = \sqrt{k}$, or $a^{\frac{1}{2}} + b^{\frac{1}{2}} = k^{\frac{1}{2}}$.

Therefore, the sum of the intercepts is $= k$, as required.

16.



$$y^2 = 4x$$
$$2t\frac{dy}{dx} = 4$$
At (1, 2), $\frac{dy}{dx} = 1$.

Therefore, the slope of the tangent line at (1, 2) is 1 and the equation of the normal is

$$\frac{y - 2}{x - 1} = -1 \text{ or } x + y = 3.$$

The centres of the two circles lie on the straight line $x + y = 3$. Let the coordinates of the centre of each circle be $(p, q) = (p, 3 - p)$. The radius of each circle is $3\sqrt{2}$. Since (1, 2) is on the circumference of the circles,

$$(p - 1)^2 + (3 - p - 2)^2 = r^2$$
$$p^2 - 2p + 1 + 1 - 2p + p^2 = (3\sqrt{2})^2$$
$$2p^2 - 4p + 2 = 18$$
$$p^2 - 2p - 8 = 0$$
$$(p - 4)(p + 2) = 0$$
$p = 4$ or $p = -2$

$q = -1$ or $q = 5$.

Therefore, the centres of the circles are $(-2, 5)$ and $(4, -1)$. The equations of the circles are

$(x + 2)^2 + (y - 5)^2 = 18$ and

$(x - 4)^2 + (y + 1)^2 = 18$.

Related Rates, pp. 569–570

1. a. $\frac{dA}{dt} = 4 \text{ m/s}^2$

b. $\frac{dS}{dt} = -3 \text{ m}^2\text{/min}$.

c. $\frac{ds}{dt} = 70$ km/h when $t = 0.25$

d. $\frac{dx}{dt} = \frac{dy}{dt}$

e. $\frac{d\theta}{dt} = \frac{\pi}{10}$ rad/s

2. $T(x) = \frac{200}{1 + x^2}$

a. $\frac{dx}{dt} = 2$ m/s

Find $\frac{dT(x)}{dt}$ when $x = 5$ m:

$$T(x) = \frac{200}{1 + x^2}$$
$$= 200(1 + x^2)^{-1}$$
$$\frac{dT(x)}{dt} = -200(1 + x^2)^{-2}2x\frac{dx}{dt}$$
$$= \frac{-400x}{(1 + x^2)^2} \cdot \frac{dx}{dt}.$$

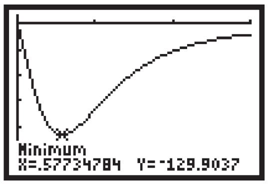
At a specific time, when $x = 5$,

$$\frac{dT(5)}{dt} = \frac{-400(5)}{(26)^2} \quad (2)$$
$$= \frac{-4000}{676}$$
$$= \frac{-1000}{169}$$
$$\frac{dT(5)}{dt} \doteq -5.9.$$

Therefore, the temperature is decreasing at a rate of 5.9 °C/s.

b. To determine the distance at which the temperature is changing fastest, graph the derivative of the function on a graphing calculator. Since the temperature decreases as the person moves away from the fire, the temperature will be changing fastest when the value of the derivative is at its minimum value.

For Y1, enter nDeriv(from the MATH menu. Then enter 200 [÷] (1+X²), X, X).



The temperature is changing fastest at about 0.58 m.

c. Solve $T''(x) = 0$.

$$T'(x) = \frac{-400x}{(1 + x^2)^2}$$
$$T'(x) = \frac{-400(1 + x^2)^2 - 2(1 + x^2)(2x)(-400x)}{(1 + x^2)^4}$$

Let $T''(x) = 0$,

$$-400(1 + x^2)^2 + 1600x^2(1 + x^2) = 0.$$

Divide,

$$400(1 + x^2) - (1 + x^2) + 4x^2 = 0$$
$$3x^2 = 1$$
$$x^2 = \frac{1}{\sqrt{3}}$$
$$x = \frac{1}{\sqrt{3}}$$

$x > 0$ or $x \doteq 0.58$.

$$y^2 = 4x$$
$$2t\frac{dy}{dx} = 4$$
At (1, 2), $\frac{dy}{dx} = 1$.

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$$(p - 1)^2 + (3 - p - 2)^2 = r^2$$
$$p^2 - 2p + 1 + 1 - 2p + p^2 = (3\sqrt{2})^2$$
$$2p^2 - 4p + 2 = 18$$
$$p^2 - 2p - 8 = 0$$
$$(p - 4)(p + 2) = 0$$
$p = 4$ or $p = -2$

$q = -1$ or $q = 5$.

Therefore, the centres of the circles are $(-2, 5)$ and $(4, -1)$. The equations of the circles are
$(x + 2)^2 + (y - 5)^2 = 18$ and
$(x - 4)^2 + (y + 1)^2 = 18$.

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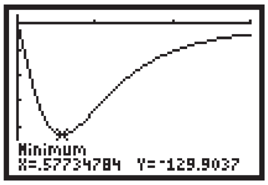
Find $\frac{dT(x)}{dt}$ when $x = 5$ m:
$$T(x) = \frac{200}{1 + x^2}$$
$$= 200(1 + x^2)^{-1}$$
$$\frac{dT(x)}{dt} = -200(1 + x^2)^{-2}2x\frac{dx}{dt}$$
$$= \frac{-400x}{(1 + x^2)^2} \cdot \frac{dx}{dt}.$$
At a specific time, when $x = 5$,
$$\frac{dT(5)}{dt} = \frac{-400(5)}{(26)^2} \quad (2)$$
$$= \frac{-4000}{676}$$
$$= \frac{-1000}{169}$$
$$\frac{dT(5)}{dt} \doteq -5.9.$$
Therefore, the temperature is decreasing at a rate of 5.9 °C/s.

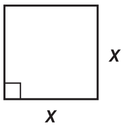
b. To determine the distance at which the temperature is changing fastest, graph the derivative of the function on a graphing calculator. Since the temperature decreases as the person moves away from the fire, the temperature will be changing fastest when the value of the derivative is at its minimum value.

For Y1, enter nDeriv(from the MATH menu. Then enter 200 [÷] (1+X²), X, X).



The temperature is changing fastest at about 0.58 m.

c. Solve $T''(x) = 0$.
$$T'(x) = \frac{-400x}{(1 + x^2)^2}$$
$$T'(x) = \frac{-400(1 + x^2)^2 - 2(1 + x^2)(2x)(-400x)}{(1 + x^2)^4}$$
Let $T''(x) = 0$,
$$-400(1 + x^2)^2 + 1600x^2(1 + x^2) = 0.$$
Divide,
$$400(1 + x^2) - (1 + x^2) + 4x^2 = 0$$
$$3x^2 = 1$$
$$x^2 = \frac{1}{\sqrt{3}}$$
$$x = \frac{1}{\sqrt{3}}$$
$x > 0$ or $x \doteq 0.58$.

3. Given square 

$\frac{dx}{dt} = 5$ cm/s.

Find $\frac{dA}{dt}$ when $x = 10$ cm.

Solution

Let the side of a square be x cm.

$A = x^2$

$\frac{dA}{dt} = 2x\frac{dx}{dt}$

At a specific time, $x = 10$ cm.

$\frac{dA}{dt} = 2(10)(5)$

$= 100$

Therefore, the area is increasing at 100 cm^2/s when a side is 10 cm.

$P = 4x$

$\frac{dP}{dt} = 4\frac{dx}{dt}$

At any time, $\frac{dx}{dt} = 5$.

$\frac{dP}{dt} = 20$.

Therefore, the perimeter is increasing at 20 cm/s.

4. Given cube with sides x cm, $\frac{dx}{dt} = 5$ cm/s.

a. Find $\frac{dv}{dt}$ when $x = 5$ cm:

$V = x^3$

$\frac{dV}{dt} = 3x^2\frac{dx}{dt}$

At a specific time, $x = 5$ cm.

$\frac{dV}{dt} = 3(5)^2(4)$

$= 300$

Therefore, the volume is increasing at 300 cm^3/s.

b. Find $\frac{dS}{dt}$ when $x = 7$ cm.

$S = 6x^2$

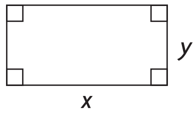
$\frac{dS}{dt} = 12x\frac{dx}{dt}$

At a specific time, $x = 7$ cm,

$\frac{dS}{dt} = 12(7)(4)$

$= 336$.

Therefore, the surface area is increasing at a rate of 336 cm^2/s.

5. Given rectangle 

$\frac{dx}{dt} = 2$ cm/s

$\frac{dy}{dt} = -3$ cm/s

Find $\frac{dA}{dt}$ when $x = 20$ cm and $y = 50$ cm.

Solution

$A = xy$

$\frac{dA}{dt} = \frac{dx}{dt}y + \frac{dy}{dt}x$

At s specific time, $x = 20$, $y = 50$,

$\frac{dA}{dt} = (2)(50) + (-3)(20)$

$= 100 - 50$

$= 40$.

Therefore, the area is increasing at a rate of 40 cm^2/s.

6. Given circle with radius r,

$\frac{dA}{dt} = -5$ m^2/s.

a. Find $\frac{dr}{dt}$ when $r = 3$ m.

$A = \pi r^2$

$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$

When $r = 3$,

$-5 = 2\pi(3)\frac{dr}{dt}$

$\frac{dr}{dt} = \frac{-5}{6\pi}$

Therefore, the radius is decreasing at a rate of $\frac{5}{6\pi}$ m/s when $r = 3$ m.

b. Find $\frac{dD}{dt}$ when $r = 3$.

$\frac{dD}{dt} = 2\frac{dr}{dt}$

$= 2\left(\frac{-5}{6\pi}\right)$

$= \frac{-5}{3\pi}$

Therefore, the diameter is decreasing at a rate of $\frac{5}{3\pi}$ m/s.

7. Given circle with radius r, $\frac{dA}{dt} = 6$ km^2/h

Find $\frac{dr}{dt}$ when $A = 9\pi$ km^2.

$$A = \pi r^2$$
$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$

When $A = 9\pi$,

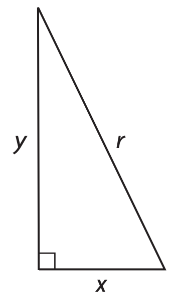
$$9\pi = \pi r^2$$
$$r^2 = 9$$
$$r = 3$$
$$r > 0.$$

When $r = 3$,

$$\frac{dA}{dt} = 6$$
$$6 = 2\pi(3)\frac{dr}{dt}$$
$$\frac{dr}{dt} = \frac{1}{\pi}.$$

Therefore, the radius is increasing at a rate of $\frac{1}{\pi}$ km/h.

8. Let x represent the distance from the wall and y the height of the ladder on the wall.



$$x^2 + y^2 = r^2$$
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2r\frac{dr}{dt}$$
$$x\frac{dx}{dt} + y\frac{dy}{dt} = r\frac{dr}{dt}$$

When $r = 5$, $y = 3$,

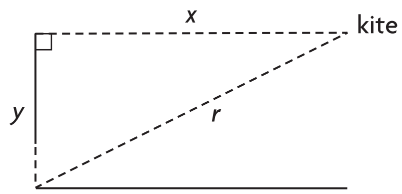
$$x^2 = 25 - 9$$
$$= 16$$
$$x = 4$$
$$x = 4, y = 3, r = 5$$
$$\frac{dx}{dt} = \frac{1}{3}, \frac{dr}{dt} = 0.$$

Substituting,

$$4\left(\frac{1}{3}\right) + 3\left(\frac{dy}{dt}\right) = 5(0)$$
$$\frac{dy}{dt} = -\frac{4}{9}.$$

Therefore, the top of the ladder is sliding down at 4 m/s.

9.

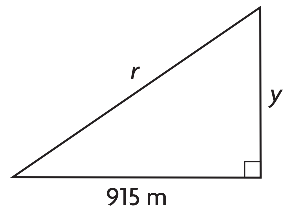


Let the variables represent the distances as shown on the diagram.

$$x^2 + y^2 = r^2$$
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2r\frac{dr}{dt}$$
$$x\frac{dx}{dt} + y\frac{dy}{dt} = r\frac{dr}{dt}$$
$$x = 30, y = 40$$
$$r^2 = 30^2 + 40^2$$
$$r = 50$$
$$\frac{dr}{dt} = ?, \frac{dx}{dt} = 10, \frac{dy}{dt} = 0$$
$$30(10) + 40(0) = 50\left(\frac{dr}{dt}\right)$$
$$\frac{dr}{dt} = 8$$

Therefore, she must let out the line at a rate of 8 m/min.

10.



Label diagram as shown.

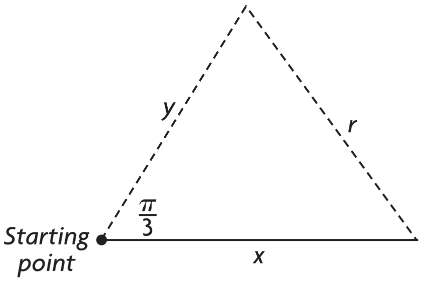
$$r^2 = y^2 + 915^2$$
$$2r\frac{dr}{dt} = 2y\frac{dy}{dt}$$
$$r\frac{dr}{dt} = y\frac{dy}{dt}$$

When $y = 1220$, $\frac{dy}{dt} = 268$ m/s.

$$r = \sqrt{1220^2 + 915^2}$$
$$= 1525$$
$$1525\left(\frac{dr}{dt}\right) = 1220 \times 268$$
$$\frac{dr}{dt} = 214 \text{ m/s}$$

Therefore, the camera–to–rocket distance is changing at 214 m/s.

11.



$\frac{dx}{dt} = 15$ km/h

$\frac{dy}{dt} = 20$ km/h

Find $\frac{dr}{dt}$ when $t = 2$ h.

Solution

Let x represent the distance cyclist 1 is from the starting point, $x \geq 0$. Let y represent the distance cyclist 2 is from the starting point, $y \geq 0$ and let r be the distance the cyclists are apart. Using the cosine law,

$$r^2 = x^2 + y^2 - 2xy\cos\frac{\pi}{3}$$
$$= x^2 + y^2 - 2xy\left(\frac{1}{2}\right)$$
$$r^2 = x^2 + y^2 - xy$$
$$2r\frac{dr}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} - \left[\frac{dx}{dt}y + \frac{dy}{dt}x\right]$$

At $t = 2$ h, $x = 30$ km, $y = 40$ km

and $r^2 = 30^2 + 40^2 - 2(30)(40)\cos\frac{\pi}{3}$
$$= 2500 - 2(1200)\left(\frac{1}{2}\right)$$
$$= 1300$$
$$r = 10\sqrt{13},\ r > 0.$$
$$\therefore 2(10\sqrt{13})\frac{dr}{dt}$$
$$= 2(30)(15) + 2(40)(20) - [15(40) + 20(30)]$$
$$20\sqrt{13}\frac{dr}{dt} = 900 + 1600 - [600 - 600]$$
$$= 1300$$
$$\frac{dr}{dt} = \frac{130}{2\sqrt{13}}$$
$$= \frac{65}{\sqrt{13}}$$
$$= \frac{65\sqrt{13}}{13}$$
$$= 5\sqrt{13}$$

Therefore, the distance between the cyclists is increasing at a rate of $5\sqrt{13}$ km/h after 2 h.

12. Given sphere $V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = 8$ cm³/s.

a. Find $\frac{dr}{dt}$ when $r = 12$ cm.

$$V = \frac{4}{3}\pi r^3$$
$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$$

At a specific time, when $r = 12$ cm:

$$8 = 4\pi(12)^2\frac{dr}{dt}$$
$$8 = 4\pi(144)\frac{dr}{dt}$$
$$\frac{1}{72\pi} = \frac{dr}{dt}.$$

Therefore, the radius is increasing at a rate of $\frac{1}{72\pi}$ cm/s.

b. Find $\frac{dr}{dt}$ when $V = 1435$ cm³.

$$V = \frac{4}{3}\pi r^3$$
$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}.$$

At a specific time, when $V = 1435$ cm³:

$$V = 1435$$
$$\frac{4}{3}\pi^3 = 1435$$
$$r^3 \doteq 342.581\,015$$
$$\doteq 6.997\,148\,6$$
$$= 7$$
$$8 \doteq 4\pi(7)^2\frac{dr}{dt}$$
$$8 = 196\pi\frac{dr}{dt}$$
$$\frac{2}{49\pi} = \frac{dr}{dt}$$
$$0.01 = \frac{dr}{dt}.$$

Therefore, the radius is increasing at $\frac{2}{49\pi}$ cm/s (or about 0.01 cm/s).

c. Find $\frac{dr}{dt}$ when $t = 33.5$ s.

When $t = 33.5$, $v = 8 \times 33.5$ cm³:

$$\frac{4}{3}\pi r^3 = 268$$
$$r^3 \doteq 63.980\,287\,12$$

$r \doteq 3.999\ 589\ 273$
$\doteq 4.$

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

At $t = 33.5$ s,

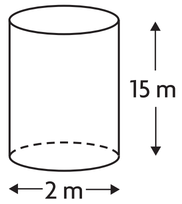
$8 \doteq 4\pi(4)^2 \frac{dr}{dt}$

$8 = 64\pi \frac{dr}{dt}$

$\frac{1}{8\pi} = \frac{dr}{dt}.$

Therefore, the radius is increasing at a rate of $\frac{1}{8\pi}$ cm/s (or 0.04 cm/s).

13. Given cylinder



$V = \pi r^2 h$

$\frac{dV}{dt} = 500$ L/min

$= 500\ 000$ cm/min

Find $\frac{dh}{dt}$.

$V = \pi r^2 h$

Since the diameter is constant at 2 m, the radius is also constant at 1 m = 100 cm.

$V = 10\ 000\pi h$

$\frac{dV}{dt} = 10\ 000\pi \frac{dh}{dt}$

$500\ 000 = 10\ 000\pi \frac{dh}{dt}$

$\frac{50}{\pi} = \frac{dh}{dt}$

Therefore, the fluid level is rising at a rate of $\frac{50}{\pi}$ cm/min.

Find t, the time of fill the cylinder.

$V = \pi r^2 h$

$V = \pi(100)^2(1\ 500)$ cm^3

$V = 150\ 000\ 00\pi$ cm^3

Since $\frac{dV}{dt} = 500\ 000$ cm^3/min, it takes $\frac{15\ 000\ 000\pi}{500\ 000}$ min,

$= 30\pi$ min to fill

$\doteq 94.25$ min.

Therefore, it will take 94.25 min, or just over 1.5 h to fill the cylindrical tank.

14. There are many possible problems.

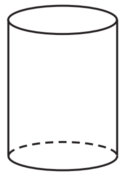
Samples:

a. The diameter of a right-circular cone is expanding at a rate of 4 cm/min. Its height remains constant at 10 cm. Find its radius when the volume is increasing at a rate of 80π cm^3/min.

b. Water is being poured into a right-circular tank at the rate of 12π m^3/min. Its height is 4 m and its radius is 1 m. At what rate is the water level rising?

c. The volume of a right-circular cone is expanding because its radius is increasing at 12 cm/min and its height is increasing at 6 cm/min. Find the rate at which its volume is changing when its radius is 20 cm and its height is 40 cm.

15. Given cylinder



$d = 1$ m

$h = 15$ m

$\frac{dr}{dt} = 0.003$ m/year

$\frac{dh}{dt} = 0.4$ m/year

Find $\frac{dV}{dt}$ at the instant $D = 1$

$V = \pi r^2 h$

$\frac{dV}{dt} = \left(2\pi r \frac{dr}{dt}\right)(h) + \left(\frac{dh}{dt}\right)(\pi r^2).$

At a specific time, when $D = 1$; i.e., $r = 0.5$,

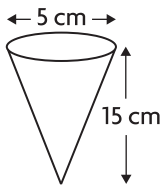
$\frac{dV}{dt} = 2\pi(0.5)(0.003)(15) + 0.4\pi(0.5)^2$

$= 0.045\pi + 0.1\pi$

$= 0.145\pi$

Therefore, the volume of the trunk is increasing at a rate of 0.145π m^3/year.

16. Given cone



$r = 5$ cm
$h = 15$ cm
$\frac{dV}{dt} = 2\text{ cm}^3/\text{min}$

Find $\frac{dh}{dt}$ when $h = 3$ cm,

$V = \frac{1}{3}\pi r^2 h.$

Using similar triangles, $\frac{r}{h} = \frac{5}{15} = \frac{1}{3}$

$r = \frac{h}{3}.$

Substituting into $V = \frac{1}{3}\pi r^2 h$,

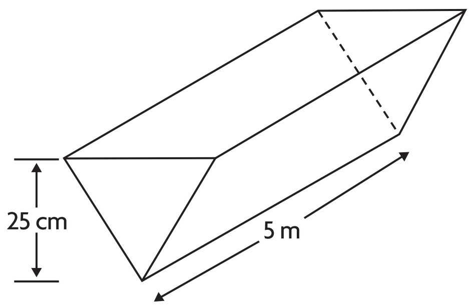
$$V = \frac{1}{3}\pi\left(\frac{h^2}{9}\right)h$$
$$= \frac{1}{27}\pi h^3$$
$$\frac{dV}{dt} = \frac{1}{9}\pi h^2 \frac{dh}{dt}$$

At a specific time, when $h = 3$ cm,

$$-2 = \frac{1}{9}\pi(3)^2 \frac{dh}{dt}$$
$$-\frac{2}{\pi} = \frac{dh}{dt}.$$

Therefore, the water level is being lowered at a rate of $\frac{2}{\pi}$ cm/min when height is 3 cm.

17. Given trough



$\frac{dV}{dt} = 0.25\frac{\text{m}^3}{\text{min}}$

Find $\frac{dh}{dt}$ when $h = 10$ cm
$= 0.1$ m.

Since the cross section is equilateral, $V = \frac{h^2}{\sqrt{3}} \times l.$

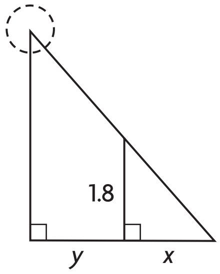
$$V = \frac{h^2}{\sqrt{3}} \times 5.$$
$$\frac{dV}{dt} = \frac{10}{\sqrt{3}}h\frac{dh}{dt}$$

At a specific, time when $h = 0.1 = \frac{1}{10}$,

$$0.25 = \frac{10}{\sqrt{3}}\frac{1}{10}\frac{dh}{dt}$$
$$0.25\sqrt{3} = \frac{dh}{dt}$$
$$\frac{\sqrt{3}}{4} = \frac{dh}{dt}$$

Therefore, the water level is rising at a rate of $\frac{\sqrt{3}}{4}$ m/min.

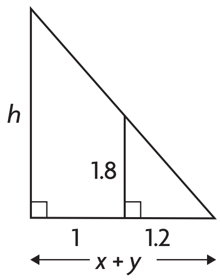
18.



$\frac{dy}{dt} = 120$ m/min

Find $\frac{dx}{dt}$ when $t = 5$ s.

Let x represent the length of the shadow. Let y represent the distance the man is from the base of the lamppost. Let h represent the height of the lamppost. At a specific instant, we have



Using similar triangles,

$$\frac{x + y}{h} = \frac{1.2}{1.8}$$
$$\frac{2.2}{h} = \frac{2}{3}$$
$$2h = 6.6$$
$$h = 3.3$$

Therefore, the lamppost is 3.3 m high.

At any time,

$$\frac{x+y}{x} = \frac{3.3}{1.8}$$

$$\frac{x+y}{x} = \frac{11}{6}$$

$$6x + 6y = 11x$$

$$6y = 5x$$

$$6\frac{dy}{dt} = 5\frac{dx}{dt}.$$

At a specific time, when $t = 5$ seconds

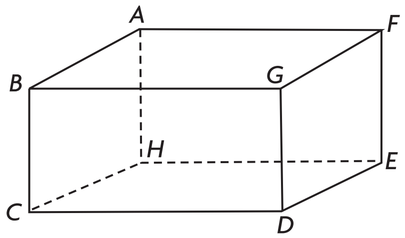
$\frac{dy}{dt} = 120$ m/min,

$$6 \times 120 = 5\frac{dx}{dt}$$

$$\frac{dx}{dt} = 144.$$

Therefore, the man's shadow is lengthening at a rate of 144 m/min after 5 s.

19. This question is similar to finding the rate of change of the length of the diagonal of a rectangular prism.



$$20\text{m} = \frac{20}{1000}\text{ km}$$

$$= \frac{1}{50}\text{ km}$$

Find $\frac{d(GH)}{dt}$ *at* $t = 10$ s,

$$= \frac{1}{360}\text{ h.}$$

Let BG be the path of the train and CH be the path of the boat:

$$\frac{d(BG)}{dt} = 60 \text{ km/h and } \frac{d(CH)}{dt} = 20 \text{ km/h.}$$

$$\text{At } t = \frac{1}{360}\text{ h, } BG = 60 \times \frac{1}{360}$$

$$= \frac{1}{6}\text{ km}$$

$$\text{and } CH = 20 \times \frac{1}{360}$$

$$= \frac{1}{18}\text{ km.}$$

Using the Pythagorean theorem,

$GH^2 = HD^2 + DG^2$ and $HD^2 = CD^2 + CH^2$

$GH^2 = CD^2 + CH^2 + DG^2$

Since $BG = CD$ and $FE = GD = \frac{1}{50}$, it follows that

$$GH^2 = BG^2 + CH^2 + \frac{1}{2500}.$$

$$2(GH)\frac{d(GH)}{dt} = 2(BG)\frac{d(BG)}{dt} + 2(CH)\frac{d(CH)}{dt}$$

At $t = 10$ s,

$$GH\frac{d(GH)}{dt} = \frac{1}{6}(60) + \frac{1}{18}(20)$$

$$\frac{\sqrt{6331}}{450}\,\frac{d(GH)}{dt} = \frac{100}{9}$$

$$\frac{d(GH)}{dt} = \frac{45000}{9\sqrt{6331}}$$

$$\doteq 62.8.$$

$$\text{And } GH^2 = \left(\frac{1}{6}\right)^2 + \left(\frac{1}{18}\right)^2 + \left(\frac{1}{50}\right)^2$$

$$= \frac{1}{36} + \frac{1}{324} + \frac{1}{2500}$$

$$= \frac{911\,664}{29\,160\,000} + 8$$

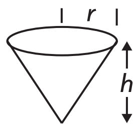
$$GH^2 = \frac{113\,958}{364\,500} + 18$$

$$= \frac{6331}{202\,500}$$

$$GH = \frac{\sqrt{6331}}{450} = \frac{\sqrt{13 \times 487}}{450}$$

Therefore, they are separating at a rate of approximately 62.8 km/h.

20. Given cone



a. $r = h$

$$\frac{dV}{dt} = 200 - 20$$

$$= 180\text{ cm}^3\text{/s}$$

Find $\frac{dh}{dt}$ when $h = 15$ cm.

$$V = \frac{1}{3}\pi r^2 h \text{ and } r = h$$

$$V = \frac{1}{3}\pi h^3.$$

$$\frac{dV}{dt} = \pi h^2\frac{dh}{dt}$$

At a specific time, $h = 15$ cm.

$$180 = \pi(15)^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{4}{5\pi}$$

Therefore, the height of the water in the funnel is increasing at a rate of $\frac{4}{5\pi}$ cm/s.

b. $\frac{dV}{dt} = 200$ cm^3/s

Find $\frac{dh}{dt}$ when $h = 25$ cm.

$$\frac{dV}{dt} = \pi h^2 \frac{dh}{dt}.$$

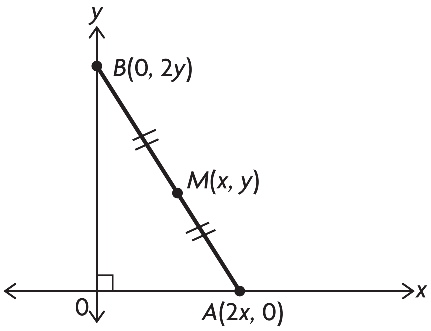
At the time when the funnel is clogged, $h = 25$ cm:

$$200 = \pi(25)^2 \frac{dh}{dt}.$$

$$\frac{dh}{dt} = \frac{8}{25\pi}.$$

Therefore, the height is increasing at $\frac{8}{25\pi}$ cm/s.

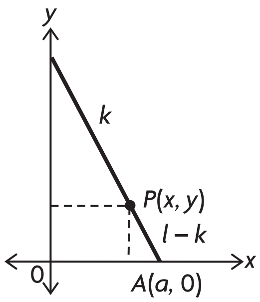
21.



Let the midpoint of the ladder be (x, y). From similar triangles, it can be shown that the top of the ladder and base of the ladder would have points $B(0, 2y)$ and $A(2x, 0)$ respectively. Since the ladder has length l,

$$(2x)^2 + (2y)^2 = l^2$$
$$4x^2 + 4y^2 = l^2$$
$$x^2 + y^2 = \frac{l^2}{4}$$
$$= \left(\frac{l}{2}\right)^2 \text{ is the required equation.}$$

Therefore, the equation of the path followed by the midpoint of the ladder represents a quarter circle with centre (0, 0) and radius $\frac{l}{2}$, with $x, y \geq 0$.



Let $P(x, y)$ be a general point on the ladder a distance k from the top of the ladder. Let $A(a, 0)$ be the point of contact of the ladder with the ground.

From similar triangles, $\frac{a}{l} = \frac{x}{k}$ or $a = \frac{xl}{k}$.

Using the Pythagorean Theorem:

$y^2 + (a - x)^2 = (l - k)^2$, and substituting $a = \frac{xl}{k}$,

$$y^2 + \left(\frac{xl}{k} - x\right)^2 = (l - k)^2$$

$$y^2 + x^2\left(\frac{l - k}{k}\right)^2 = (l - k)^2$$

$$\frac{(l - k)^2}{k^2}x^2 + y^2 = (l - k)^2$$

$$\frac{x^2}{k^2} + \frac{y^2}{(l - k)^2} = 1 \text{ is the required equation.}$$

Therefore, the equation is the first quadrant portion of an ellipse.

The Natural Logarithm and its Derivative, p. 575

1. A natural logarithm has base e; a common logarithm has base 10.

2. Since $e = \lim_{h \to 0} (1 + h)^{\frac{1}{h}}$, let $h = \frac{1}{n}$. Therefore,

$$e = \lim_{\frac{1}{n} \to 0} \left(1 + \frac{1}{n}\right)^n,$$

But as $\frac{1}{n} \to 0$, $n \to \infty$.

Therefore, $e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$.

If $n = 100$, $e \doteq \left(1 + \frac{1}{100}\right)^{100}$

$$= 1.01^{100}$$
$$\doteq 2.70481.$$

Try $n = 100\,000$, etc.

At a specific time, $h = 15$ cm.

$$180 = \pi(15)^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{4}{5\pi}$$

Therefore, the height of the water in the funnel is increasing at a rate of $\frac{4}{5\pi}$ cm/s.

b. $\frac{dV}{dt} = 200 \text{ cm}^3/\text{s}$

Find $\frac{dh}{dt}$ when $h = 25$ cm.

$$\frac{dV}{dt} = \pi h^2 \frac{dh}{dt}.$$

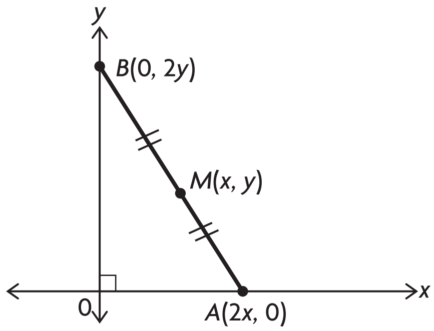
At the time when the funnel is clogged, $h = 25$ cm:

$$200 = \pi(25)^2 \frac{dh}{dt}.$$

$$\frac{dh}{dt} = \frac{8}{25\pi}.$$

Therefore, the height is increasing at $\frac{8}{25\pi}$ cm/s.

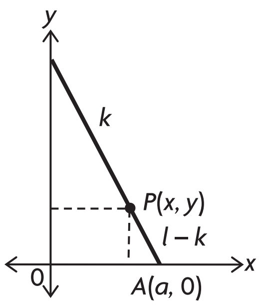
21.



Let the midpoint of the ladder be (x, y). From similar triangles, it can be shown that the top of the ladder and base of the ladder would have points $B(0, 2y)$ and $A(2x, 0)$ respectively. Since the ladder has length l,

$$\begin{aligned} (2x)^2 + (2y)^2 &= l^2 \\ 4x^2 + 4y^2 &= l^2 \\ x^2 + y^2 &= \frac{l^2}{4} \\ &= \left(\frac{l}{2}\right)^2 \text{ is the required equation.} \end{aligned}$$

Therefore, the equation of the path followed by the midpoint of the ladder represents a quarter circle with centre (0, 0) and radius $\frac{l}{2}$, with $x, y \geq 0$.



Let $P(x, y)$ be a general point on the ladder a distance k from the top of the ladder. Let $A(a, 0)$ be the point of contact of the ladder with the ground.

From similar triangles, $\frac{a}{l} = \frac{x}{k}$ or $a = \frac{xl}{k}$.

Using the Pythagorean Theorem:

$y^2 + (a - x)^2 = (l - k)^2$, and substituting $a = \frac{xl}{k}$,

$$y^2 + \left(\frac{xl}{k} - x\right)^2 = (l - k)^2$$

$$y^2 + x^2\left(\frac{l - k}{k}\right)^2 = (l - k)^2$$

$$\frac{(l - k)^2}{k^2}x^2 + y^2 = (l - k)^2$$

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$$\begin{aligned} \text{If } n = 100,\ e &\doteq \left(1 + \frac{1}{100}\right)^{100} \\ &= 1.01^{100} \\ &\doteq 2.70481. \end{aligned}$$

Try $n = 100\,000$, etc.

3. a. $\frac{dy}{dx} = \frac{d}{dx}(\ln(5x+8))$

$= \frac{1}{5x+8}\frac{d}{dx}(5x+8)$

$= \frac{5}{5x+8}$

b. $\frac{dy}{dx} = \frac{d}{dx}(\ln(x^2+1))$

$= \frac{1}{x^2+1}\frac{d}{dx}(x^2+1)$

$= \frac{2x}{x^2+1}$

c. $\frac{ds}{dt} = \frac{d}{dt}(5\ln t^3)$

$= \frac{d}{dt}(15\ln t)$

$= \frac{15}{t}$

d. $\frac{dy}{dx} = \frac{d}{dx}(5\ln\sqrt{x+1})$

$= \frac{d}{dx}(\ln(x+1)^{\frac{1}{2}})$

$= \frac{d}{dx}\left(\frac{1}{2}\ln(x+1)\right)$

$= \frac{1}{2(x+1)}$

e. $\frac{ds}{dt} = \frac{d}{dt}(\ln(t^3-2t^2+5))$

$= \frac{1}{t^3-2t^2+5}\frac{d}{dx}(t^3-2t^2+5)$

$= \frac{3t^2-4t}{t^3-2t^2+5}$

f. $\frac{dw}{dz} = \frac{d}{dz}(5\ln\sqrt{z^2+3z})$

$= \frac{d}{dz}(\ln(z^2+3z)^{\frac{1}{2}})$

$= \frac{d}{dz}\left(\frac{1}{2}\ln(z^2+3z)\right)$

$= \frac{1}{2(z^2+3z)}\frac{d}{dz}(z^2+3z)$

$= \frac{2z+3}{2(z^2+3z)}$

4. a. $f'(x) = \frac{d}{dx}(x\ln x)$

$= (1)\ln x + x\left(\frac{1}{x}\right)$

$= \ln x + 1$

b. Since e^x and $\ln x$ are inverse functions, the composite function $y = e^{\ln x}$ is equivalent to $y = x$.

So $\frac{dy}{dx} = \frac{d}{dx}(x) = 1$.

c. $\frac{dv}{dt} = \frac{d}{dt}(e^t\ln t)$

$= (e^t)\ln t + e^t\left(\frac{1}{t}\right)$

$= e^t\ln t + \frac{e^t}{t}$

d. $g(z) = \ln(e^{-z} + ze^{-z})$

$g'(z) = \frac{1}{e^{-z}+ze^{-z}}[-e^{-z} + (e^{-z} - ze^{-z})]$

$= \frac{-ze^{-z}}{e^{-z}+ze^{-z}}$

e. $\frac{ds}{dt} = \frac{d}{dt}\left(\frac{e^t}{\ln t}\right)$

$= \frac{\ln t(e^t) - e^t\left(\frac{1}{t}\right)}{(\ln t)^2}$

$= \frac{te^t\ln t - e^t}{t(\ln t)^2}$

f. $h(u) = e^{\sqrt{a}}\ln u^{\frac{1}{2}}$

$= e^{\sqrt{a}}\left(\frac{1}{2}\ln u\right)$

$h'(u) = e^{\sqrt{a}}\left(\frac{1}{2\sqrt{u}}\right)\left(\frac{1}{2}\ln u\right) + \frac{1}{2}\left(\frac{1}{u}\right)e^{\sqrt{a}}$

$= \frac{1}{2}e^{\sqrt{a}}\left(\frac{1}{2}e^{1\sqrt{a}}\ln u + \frac{1}{u}\right)$

5. a. $g(x) = e^{2x-1}\ln(2x-1)$

$g'(x) = e^{2x-1}(2)\ln(2x-1) + \left(\frac{1}{2x-1}\right)(2)e^{2x-1}$

$g'(1) = e^2(2)\ln(1) + 1(2)e^1$

$= 2e$

b. $f(t) = \ln\left(\frac{t-1}{3t+5}\right)$

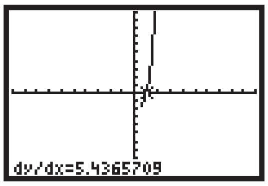
$f'(t) = \left(\frac{3t+t}{t-1}\right)\left[\frac{3t+5-3(t-1)}{(3t+5)^2}\right]$

$f'(5) = \frac{20}{4}\left[\frac{20-12}{20^2}\right]$

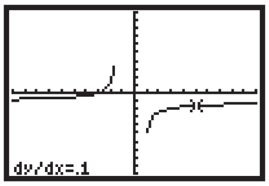
$= \frac{8}{4\times 20}$

$= \frac{1}{10}$

$= 0.1$

c. 

The value shown is approximately $2e$, which matches the calculation in part a.



This value matches the calculation in part b.

6. a. $f(x) = \ln(x^2 + 1)$

$$f'(x) = \left(\frac{1}{1 + x^2}\right)(2x)$$
$$= \frac{2x}{1 + x^2}$$

Since $1 + x^2 > 0$ for all x, $f'(x) = 0$ when $2x = 0$, i.e., when $x = 0$.

b. $f(x) = (\ln x + 2x)^{\frac{1}{3}}$

$$f'(x) = \frac{1}{3}(\ln x + 2x)^{-\frac{2}{3}}\left(\frac{1}{x} + 2\right)$$
$$= \frac{\frac{1}{x} + 2}{3(\ln x + 2x)^{\frac{2}{3}}}$$

$f'(x) = 0$ if $\frac{1}{x} + 2 = 0$ and $(\ln x + 2x)^{\frac{2}{3}} \neq 0$.

$\frac{1}{x} + 2 = 0$ when $x = -\frac{1}{2}$.

Since $f(x)$ is defined only for $x > 0$, there is no solution to $f'(x) = 0$.

c. $f(x) = (x^2 + 1)^{-1}\ln(x^2 + 1)$

$$f'(x) = -(x^2 + 1)^{-2}(2x)\ln(x^2 + 1) + (x^2 + 1)^{-1}\left(\frac{2x}{x^2 + 1}\right)$$
$$= \frac{2x(1 - \ln(x^2 + 1))}{(x^2 + 1)^2}$$

Since $(x^2 + 1)^2 \geq 1$ for all x, $f'(x) = 0$, when $2x(1 - \ln(x^2 + 1)) = 0$.

Hence, the solution is

$x = 0$ or $\ln(x^2 + 1) = 1$
$$x^2 + 1 = e$$
$$x = \pm\sqrt{e - 1}.$$

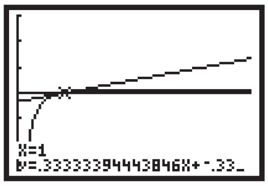
7. a. $f(x) = \frac{\ln\sqrt[3]{x}}{x}$
$$= \frac{\frac{1}{3}\ln x}{x}$$

$$f'(x) = \frac{1}{3}\left[\frac{\frac{1}{x}\cdot x - \ln x}{x^2}\right]$$

At the point (1, 0), the slope of the tangent line is

$$f'(1) = \frac{1}{3}\left[\frac{1 - 0}{1}\right]$$
$$= \frac{1}{3}.$$

The equation of the tangent line is $y = \frac{1}{3}(x - 1)$ or $x - 3y - 1 = 0$.

b. 

c. The equation on the calculator is in a different form, but is equivalent to the equation in part a.

8. The line defined by $3x - 6y - 1 = 0$ has slope $\frac{1}{2}$.

For $y = \ln x - 1$, the slope at any point is $\frac{dy}{dx} = \frac{1}{2}$.

Therefore, at the point of tangency $\frac{1}{x} = \frac{1}{2}$, or $x = 2$ and $y = \ln 2 - 1$.

The equation of the tangent is

$y - (\ln 2 - 1) = \frac{1}{2}(x - 2)$ or

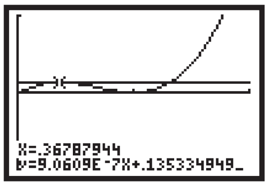
$x - 2y + (2\ln 2 - 4) = 0$

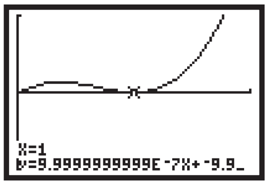
9. a. For a horizontal tangent line, the slope equals 0. We solve:

$$f'(x) = 2(x\ln x)\left(\ln x + x\cdot\frac{1}{x}\right) = 0$$

$x = 0$ or	$\ln x = 0$ or	$\ln x = -1$
No ln in the domain	$x = 1$	$x = e^{-1} = \frac{1}{e}$.

The points on the graph of $f(x)$ at which there are horizontal tangents are $\left(\frac{1}{e}, \frac{1}{e^2}\right)$ and (1, 0).

b. 



c. The solution in part a. is more precise and efficient.

10. $\frac{dy}{dx} = \frac{d}{dx}(\ln(1 + e^{-x}))$

$= \frac{1}{1 + e^{-x}}\frac{d}{dx}(1 + e^{-x})$

$= \frac{-e^{-x}}{1 + e^{-x}}$

At $x = 0$, $y = \ln(1 + e^{-0}) = \ln 2$ and $\frac{dy}{dx} = \frac{-e^{-0}}{1 + e^{-0}} = -\frac{1}{2}$, so the equation of the tangent line at this point is $y = -\frac{1}{2}x + \ln 2$

11. $v(t) = 90 - 30\ln(3t + 1)$

a. At $t = 0$, $v(0) = 90 - 30\ln(1) = 90$ km/h.

b. $a = v'(t) = \frac{-30}{3t + 1} \cdot 3 = \frac{-90}{3t + 1}$

c. At $t = 2$, $a = -\frac{90}{7} \doteq -12.8$ km/h/s.

d. The car is at rest when $v = 0$.

We solve:

$$v(t) = 90 - 30\ln(3t + 1) = 0$$
$$\ln(3t + 1) = 3$$
$$3t + 1 = e^3$$
$$t = \frac{e^3 - 1}{3} = 6.36 \text{ s.}$$

12. By definition, $\frac{d}{dx}\ln x = \lim_{h \to 0} \frac{\ln(x + h) - \ln x}{h}$

$= \frac{1}{x}$.

The derivative of $\ln x$ at $x = 2$ is

$$\lim_{h \to 0} \frac{\ln(2 + h) - \ln 2}{h} = \frac{1}{2}.$$

13. a. $f'(x) = \frac{d}{dx}(\ln(\ln x))$

$= \frac{1}{\ln x}\frac{d}{dx}(\ln x)$

$= \frac{1}{x\ln x}$

b. The natural logarithm is defined for $x > 0$, so $f(x) = \ln(\ln x)$ is defined for $\ln x > 0$, which is satisfied for $x > 1$. The function's domain is therefore $\{x \in \mathbf{R} | x > 1\}$.

The derivative $f'(x) = \frac{1}{x\ln x}$ is defined where $\ln x$ is defined, which is for $x > 0$ and where $x\ln x \neq 0$, which is for $x \neq 0$ and $x \neq 1$. The domain of the derivative is therefore $\{x \in \mathbf{R} | x > 0 \text{ and } x \neq 1\}$.

The Derivatives of General Logarithmic Functions, p. 578

1. a. $y = \log_5 x$

$\frac{dy}{dx} = \frac{1}{x\ln 5}$

b. $y = \log_3 x$

$\frac{dy}{dx} = \frac{1}{x\ln 3}$

c. $y = 2\log_4 x$

$\frac{dy}{dx} = \frac{2}{x\ln 4}$

d. $y = -3\log_7 x$

$\frac{dy}{dx} = \frac{-3}{x\ln 7}$

e. $y = -(\log x)$

$\frac{dy}{dx} = \frac{-1}{x\ln 10}$

f. $y = 3\log_6 x$

$\frac{dy}{dx} = \frac{3}{x\ln 6}$

2. a. $y = \log_3(x + 2)$

$\frac{dy}{dx} = \frac{f'(x)}{f(x)\ln(a)}$

$= \frac{1}{(x + 2)\ln 3}$

b. $y = \log_8(2x)$

$\frac{dy}{dx} = \frac{f'(x)}{f(x)\ln(a)}$

$= \frac{2}{(2x)\ln 8} = \frac{1}{x\ln 8}$

c. $y = -3\log_3(2x + 3)$

$\frac{dy}{dx} = \frac{f'(x)}{f(x)\ln(a)}$

$= \frac{-6}{(2x + 3)\ln 3}$

d. $y = \log_{10}(5 - 2x)$

$\frac{dy}{dx} = \frac{f'(x)}{f(x)\ln(a)}$

$= \frac{-2}{(5 - 2x)\ln 10}$

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10. $\frac{dy}{dx} = \frac{d}{dx}(\ln(1 + e^{-x}))$

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$\frac{dy}{dx} = \frac{f'(x)}{f(x)\ln(a)}$

$= \frac{2}{(2x + 6)\ln 8} = \frac{1}{(x + 3)\ln 8}$

f. $y = \log_7(x^2 + x + 1)$

$$\frac{dy}{dx} = \frac{f'(x)}{f(x)\ln(a)}$$

$$= \frac{2x + 1}{(x^2 + x + 1)\ln 7}$$

3. a. $f(t) = \log_2 \frac{t + 1}{2t + 7}$

$$f(t) = \log_2(t + 1) - \log_2(2t + 7)$$

$$f'(t) = \frac{1}{(t + 1)\ln 2} - \frac{2}{(2t + 7)\ln 2}$$

$$f'(3) = \frac{1}{4\ln 2} - \frac{2}{13\ln 2}$$

$$= \frac{13}{52\ln 2} - \frac{8}{52\ln 2}$$

$$= \frac{5}{52\ln 2}$$

b. $h(t) = \log_3(\log_2(t))$

$$h'(t) = \frac{f'(t)}{f(t)\ln a}$$

$$= \frac{\frac{1}{t\ln 2}}{\log_2(t)\ln 3}$$

$$= \frac{1}{t\log_2(t)(\ln 3)(\ln 2)}$$

$$h'(8) = \frac{1}{8\log_2(8)(\ln 3)(\ln 2)}$$

4. a. $y = \log_{10}\frac{1 + x}{1 - x}$

$$y = \log_{10}(1 + x) - \log_{10}(1 - x)$$

$$\frac{dy}{dx} = \frac{1}{(1 + x)\ln 10} - \frac{-1}{(1 - x)\ln 10}$$

$$= \frac{1}{(1 + x)\ln 10} + \frac{1}{(1 - x)\ln 10}$$

$$= \frac{1 - x}{(1 - x^2)\ln 10} + \frac{1 + x}{(1 - x^2)\ln 10}$$

$$= \frac{2}{(1 - x^2)\ln 10}$$

b. $y = \log_2\sqrt{x^2 + 3x}$

$$y = \frac{1}{2}\log_2(x^2 + 3x)$$

$$\frac{dy}{dx} = \frac{2x + 3}{2(x^2 + 3x)\ln(2)}$$

c. $y = 2\log_3(5^x) - \log_3(4^x)$

$$y = 2x\log_3 5 - x\log_3 4$$

$$\frac{dy}{dx} = 2\log_3 5 - \log_3 4$$

$$= \frac{2\ln 5}{\ln 3} - \frac{\ln 4}{\ln 3}$$

$$\frac{dy}{dx} = \frac{2\ln 5 - \ln 4}{\ln 3}$$

d. $y = 3^x\log_3 x$

$$\frac{dy}{dx} = \ln 3(3^x)(\log_3 x) + (3^x)\frac{1}{x\ln 3}$$

$$= \ln 3(3^x)\frac{\ln x}{\ln 3} + (3^x)\frac{1}{x\ln 3}$$

$$= \frac{x\ln 3(3^x)(\ln x) + 3^x}{x\ln 3}$$

e. $y = 2x\log_4(x)$

$$\frac{dy}{dx} = 2\log_4(x) + (2x)\frac{1}{x\ln 4}$$

$$= 2\frac{\ln x}{\ln 4} + \frac{2}{\ln 4}$$

$$= \frac{2\ln x + 2}{\ln 4}$$

$$= \frac{2\ln x + 2}{2\ln 2}$$

$$= \frac{\ln x + 1}{\ln 2}$$

f. $y = \frac{\log_5(3x^2)}{\sqrt{x + 1}}$

$$\frac{dy}{dx} = \frac{\sqrt{x + 1}\,\frac{6x}{3x^2\ln 5} - \log_5(3x^2)\frac{1}{2\sqrt{x + 1}}}{x + 1}$$

$$= \frac{\frac{2\sqrt{x + 1}}{x\ln 5} - \frac{\log_5(3x^2)}{2\sqrt{x + 1}}}{x + 1}$$

$$= \frac{\frac{4x + 1}{2x\ln 5\sqrt{x + 1}} - \frac{\log_5(3x^2)x\ln 5}{2\sqrt{x + 1}\,x\ln 5}}{x + 1}$$

$$= \frac{\frac{4x + 1 - \frac{\ln(3x^2)}{\ln 5}x\ln 5}{2x\ln 5\sqrt{x + 1}}}{x + 1}$$

$$= \frac{4x + 1 - x\ln(3x^2)}{2x\ln 5(x + 1)^{\frac{3}{2}}}$$

5. $y = x\log x$

$$\frac{dy}{dx} = \log x + (x)\frac{1}{x\ln 10}$$

$$= \log x + \frac{1}{\ln 10}$$

$$= \frac{\ln x}{\ln 10} + \frac{1}{\ln 10}$$

$$= \frac{\ln x + 1}{\ln 10}$$

To find the slope of the tangent line, find $y'(10)$:

$\frac{\ln(10) + 1}{\ln 10} = 1.434$

We also need to find a point on this line to determine its equation:

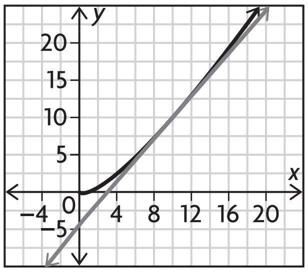
$y = 10\log(10)$

$y = 10$

So the point (10, 10) is on the tangent line.

$y - 10 = 1.434(x - 10)$

$y = 1.434x - 4.343$



6. $y = \log_a kx$

$$\frac{dy}{dx} = \frac{f'(x)}{f(x)\ln(a)} = \frac{k}{kx\ln(a)} = \frac{1}{x\ln(a)} = \frac{1}{x\ln(a)}$$

7. $y = 10^{2x-9}\log_{10}(x^2 - 3x)$

$$\frac{dy}{dx} = 2\ln 10(10^{2x-9})\log_{10}(x^2 - 3x) + \frac{2x - 3}{(x^2 - 3x)\ln 10}(10^{2x-9})$$

At $x = 5$,

$$\frac{dy}{dx} = 2\ln 10(10^{2(5)-9})\log_{10}((10)^2 - 3(10)) + \frac{2(10) - 3}{((10)^2 - 3(10))\ln 10}(10^{2(10)-9})$$

$$\frac{dy}{dx} = 49.1$$

The slope of the tangent line is 49.1 at the point (5, 10), so the equation of the line is

$y - 10 = 49.1(x - 5)$

$y = 49.1x - 235.5$

8. $s(t) = t\log_6(t + 1)$

To find whether the function is increasing or decreasing at $t = 15$, we need to find the derivative and evaluate it at $t = 15$.

$$s'(t) = \log_6(t + 1) + (t)\frac{1}{(t + 1)\ln 6} = \log_6(t) + \frac{t}{(t + 1)\ln 6}$$

$$s'(15) = \log_6(16) + \frac{15}{16\ln 6} \doteq 2.14$$

Since the derivative is positive at $t = 15$, the distance is increasing at that point.

9. a. $y = \log_3 x$

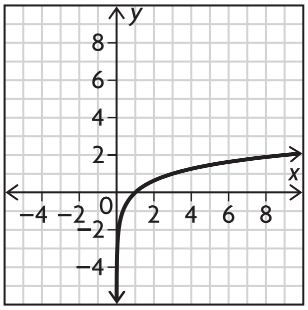
$$\frac{dy}{dx} = \frac{1}{x\ln 3}$$

To find the slope of the tangent line, let $x = 9$

$$m = \frac{1}{9\ln 3} = 0.1$$

The equation of the tangent line is $y = 0.1x + 1.1$.

b.



Vertical asymptote at $x = 0$

c. The tangent line will intersect this asymptote because it is defined for $x = 0$.

10. $f(x) = \ln(x^2 - 4)$

$\ln(x)$ is defined for $x > 0$, so $\ln(x^2 - 4)$ is defined for $x^2 - 4 > 0$. Therefore, the domain is $\{x \in \mathbf{R} | x < 2 \text{ or } x > 2\}$

$$f'(x) = \frac{2x}{x^2 - 4}$$

Let $f'(x) = 0$:

$$\frac{2x}{x^2 - 4} = 0$$

Therefore, $2x = 0$

$x = 0$

There is a critical number at $x = 0$

Since the derivative is undefined for $x = \pm 2$, there are also critical numbers at $x = 2$ and $x = -2$. However, since the function is undefined for $-2 < x < 2$, we do not need to include $x = 0$ as a critical number.

x	$x < -2$	-2	$-2 < x < 2$	2	$x > 2$
$\frac{dy}{dx}$	$-$	DNE		DNE	$+$
Graph	Dec		Undefined		Inc

Therefore the function is decreasing for $x < -2$ and increasing for $x > 2$.

11. a. $y = x \ln x$

To determine whether this function has points of inflection, we need to look at the second derivative.

$$\frac{dy}{dx} = \ln x + (x)\frac{1}{x}$$
$$= \ln x + 1$$
$$\frac{d^2y}{dx^2} = \frac{1}{x}$$

Let $\frac{d^2y}{dx^2} = 0$:

$$\frac{1}{x} = 0$$

This has no solution, but $\frac{1}{x}$ is undefined for $x = 0$, $x = 0$ is a possible point of inflection.

x	$x < 0$	0	$x > 0$
$\frac{d^2y}{dx^2}$		DNE	$+$
Graph	C. Down	Point of Inflection	C. Up

The graph switches from concave down to concave up at $x = 0$, so there is a point of inflection at $x = 0$.

b. $y = 3 - 2 \log x$

To determine whether this function has points of inflection, we need to look at the second derivative.

$$\frac{dy}{dx} = \frac{-2}{x \ln 10}$$
$$\frac{d^2y}{dx^2} = \frac{2}{x^2 \ln 10}$$

Let $\frac{d^2y}{dx^2} = 0$:

$$\frac{2}{x^2 \ln 10} = 0$$

This has no solution, but $\frac{2}{x^2 \ln 10}$ is undefined for $x = 0$, so $x = 0$ is a possible point of inflection.

x	$x < 0$	0	$x > 0$
$\frac{d^2y}{dx^2}$	$+$	DNE	$+$
Graph	C. Up		C. Up

Since the graph is always concave up, there is no point of inflection.

12. $y = 3^x$

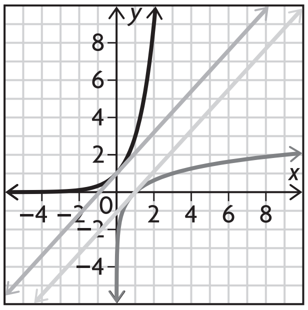
$$\frac{dy}{dx} = 3^x \ln 3$$
$$m = 3^{(0)} \ln 3$$
$$= \ln 3$$

So the slope of $y = 3^x$ at $(0, 1)$ is ln3.

$$y = \log_3 x$$
$$\frac{dy}{dx} = \frac{1}{x \ln 3}$$
$$m = \frac{1}{(1)\ln 3}$$
$$= \frac{1}{\ln 3}$$

So the slope of $y = \log_3 x$ at $(1, 0)$ is $\frac{1}{\ln 3}$.

Since $\ln 3 > 1$, the slope of $y = 3^x$ at $(0, 1)$ is greater than the slope of $y = \log_3 x$ at $(1, 0)$



Logarithmic Differentiation, p. 582

1. a. $y = x^{\sqrt{10}} - 3$

$$\frac{dy}{dx} = \sqrt{10}x^{\sqrt{10}-1}$$

b. $f(x) = 5x^{3\sqrt{2}}$

$$f'(x) = 15\sqrt{2}x^{3\sqrt{2}-1}$$

c. $s = t^{\pi}$

$$\frac{ds}{dt} = \pi t^{\pi - 1}$$

d. $f(x) = x^e + e^x$

$$f'(x) = ex^{e-1} + e^x$$

x	$x < -2$	-2	$-2 < x < 2$	2	$x > 2$
$\frac{dy}{dx}$	$-$	DNE		DNE	+
Graph	Dec		Undefined		Inc

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$\frac{d^2y}{dx^2}$		DNE	+
Graph	C. Down	Point of Inflection	C. Up

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To determine whether this function has points of inflection, we need to look at the second derivative.

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Let $\frac{d^2y}{dx^2} = 0$:

$$\frac{2}{x^2 \ln 10} = 0$$

This has no solution, but $\frac{2}{x^2 \ln 10}$ is undefined for $x = 0$, so $x = 0$ is a possible point of inflection.

x	$x < 0$	0	$x > 0$
$\frac{d^2y}{dx^2}$	+	DNE	+
Graph	C. Up		C. Up

Since the graph is always concave up, there is no point of inflection.

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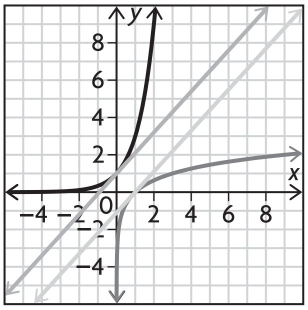

Logarithmic Differentiation, p. 582

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c. $s = t^{\pi}$

$$\frac{ds}{dt} = \pi t^{\pi - 1}$$

d. $f(x) = x^e + e^x$

$$f'(x) = ex^{e-1} + e^x$$

2. a. $y = x^{\ln x}$

$\ln y = \ln (x^{\ln x})$

$\ln y = \ln x \ln x$

$\ln y = (\ln x)^2$

$$\frac{1}{y}\frac{dy}{dx} = 2\ln x\left(\frac{1}{x}\right)$$

$$\frac{dy}{dx} = \frac{2y\ln x}{x}$$

$$= \frac{2x^{\ln x}\ln x}{x}$$

b. $$y = \frac{(x+1)(x-3)^2}{(x+2)^3}$$

$$\ln y = \ln \frac{(x+1)(x-3)^2}{(x+2)^3}$$

$$\ln y = \ln (x+1) + \ln (x-3)^2 - \ln (x+2)^3$$

$$\ln y = \ln (x+1) + 2\ln (x-3) - 3\ln (x+2)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x+1} + \frac{2}{x-3} - \frac{3}{x+2}$$

$$\frac{dy}{dx} = y\left(\frac{1}{x+1} + \frac{2}{x-3} - \frac{3}{x+2}\right)$$

$$= \frac{(x+1)(x-3)^2}{(x+2)^3}\left(\frac{1}{x+1} + \frac{2}{x-3} - \frac{3}{x+2}\right)$$

c. $y = x^{\sqrt{x}}$

$\ln y = \sqrt{x}\ln x$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2\sqrt{x}}\ln x + \sqrt{x}\frac{1}{x}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{\ln x + 2}{2\sqrt{x}}$$

$$\frac{dy}{dx} = (y)\frac{\ln x + 2}{2\sqrt{x}}$$

$$\frac{dy}{dx} = (x^{\sqrt{x}})\frac{\ln x + 2}{2\sqrt{x}}$$

d. $$s = \left(\frac{1}{t}\right)^t$$

$$\ln s = t\ln\frac{1}{t}$$

$$\frac{1}{s}\frac{ds}{dt} = \ln\frac{1}{t} + t\left(\frac{1}{\frac{1}{t}}\right)\left(\frac{-1}{t^2}\right)$$

$$\frac{1}{s}\frac{ds}{dt} = \ln\frac{1}{t} + t(t)\left(\frac{-1}{t^2}\right)$$

$$\frac{1}{s}\frac{ds}{dt} = \ln\frac{1}{t} - 1$$

$$\frac{ds}{dt} = s\left(\ln\frac{1}{t} - 1\right)$$

$$\frac{ds}{dt} = \left(\frac{1}{t}\right)^t\left(\ln\frac{1}{t} - 1\right)$$

3. a. $y = f(x) = x^{-x}$

$\ln y = x\ln x$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \ln x + x\left(\frac{1}{x}\right)$$

$$\frac{dy}{dx} = x^{-x}(\ln x + 1)$$

$f'(e) = e^e(\ln e + 1) = 2e^e$

b. $s = e^t + t^e$

$$\frac{ds}{dt} = e^t + et^{e-1}$$

When $t = 2$, $\frac{ds}{dt} = e^2 + e\cdot 2^{e-1}$

c. $$y = \frac{(x-3)^2\sqrt[3]{x+1}}{(x-4)^5}$$

Let $y = f(x)$

$$\ln y = 2\ln (x-3) + \frac{1}{3}\ln (x+1) - 5\ln (x-4)$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{2}{x-3} + \frac{1}{3(x+1)} - \frac{5}{x-4}$$

$$\frac{dy}{dx} = y\left(\frac{2}{x-3} + \frac{1}{3(x+1)} - \frac{5}{x-4}\right)$$

$$f'(7) = f(7)\left(\frac{2}{4} + \frac{1}{24} - \frac{5}{3}\right)$$

$$= \frac{32}{243}\left(-\frac{27}{24}\right) = -\frac{4}{27}$$

4. $y = x(x^2)$

The point of contact is $(2, 16)$. The slope of the tangent line at any point on the curve is given by $\frac{dy}{dx}$.

We take the natural logarithm of both sides and differentiate implicitly with respect to x.

$y = x^{(x^2)}$

$\ln y = x^2\ln x$

$$\frac{1}{y}\cdot\frac{dy}{dx} = 2x\ln x + x$$

At the point (2, 16), $\frac{dy}{dx} = 16(4\ln 2 + 2)$.

The equation of the tangent line at (2, 16) is

$y - 16 = 32(2\ln 2 + 1)(x - 2)$ or

$y = 32(2\ln 2 + 1)x - 128\ln 2 - 48.$

5. $y = \frac{1}{(x+1)(x+2)(x+3)}$

We take the natural logarithm of both sides and differentiate implicitly with respect to x to find $\frac{dy}{dx}$, the slope of the tangent line.

$$\ln y = \ln(1) - \ln(x+1) - \ln(x+2) - \ln(x+3)$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = -\frac{1}{x+1} - \frac{1}{x+2} - \frac{1}{x+3}$$

The point of contact is $(0, \frac{1}{6})$.

At $\left(0, \frac{1}{6}\right)$,

$$\frac{dy}{dx} = \frac{1}{6}\left(-1 - \frac{1}{2} - \frac{1}{3}\right) = \frac{1}{6}\left(-\frac{11}{6}\right) = -\frac{11}{36}.$$

6. $y = x^{\frac{1}{x}}, x > 0$

We take the natural logarithm of both sides and differentiate implicitly with respect to x to find $\frac{dy}{dx}$. the slope of the tangent.

$$y = x^{\frac{1}{x}}$$

$$\ln y = \frac{1}{x}\ln x = \frac{\ln x}{x}$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{\left(\frac{1}{x}\right)(x) - (\ln x)(1)}{x^2}$$

$$\frac{dy}{dx} = \frac{x^{\frac{1}{x}}(1 - \ln x)}{}$$

We want the values of x so that $\frac{dy}{dx} = 0$.

$$\frac{x^{\frac{1}{x}}(1 - \ln x)}{} = 0$$

Since $x^{\frac{1}{x}} \neq 0$ and $x^2 > 0$, we have $1 - \ln x = 0$

$$\ln x = 1$$

$$x = e.$$

The slope of the tangent is 0 at $(e, e^{\frac{1}{e}})$.

7. We want to determine the points on the given curve at which the tangent lines have slope 6. The slope of the tangent at any point on the curve is given by

$$\frac{dy}{dx} = 2x + \frac{4}{x}.$$

To find the required points, we solve:

$$2x + \frac{4}{x} = 6$$

$$x^2 - 3x + 2 = 0$$

$$(x-1)(x-2) = 0$$

$$x = 1 \text{ or } x = 2.$$

The tangents to the given curve at (1, 1) and (2, 4 + 4 ln 2) have slope 6.

8. We first must find the equation of the tangent at A(4, 16). We need $\frac{dy}{dx}$ for $y = x^{\sqrt{x}}$.

$$\ln y = \sqrt{x}\ln x$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{x}}\ln x + \sqrt{x}\cdot\frac{1}{x}$$

$$= \frac{\ln x + 2}{2\sqrt{x}}$$

At (4, 16), $\frac{dy}{dx} = 16\,\frac{(\ln 4 + 2)}{4} = 4\ln 4 + 8.$

The equation of the tangent is

$y - 16 = (4\ln 4 + 8)(x - 4).$

The y-intercept is $-16(\ln 4 + 1)$.

The x-intercept is $\frac{-4}{\ln 4 + 2} + 4 = \frac{4\ln 4 + 4}{\ln 4 + 2}$.

The area of $\triangle OBC$ is $\frac{1}{2}\left(\frac{4\ln 4 + 4}{\ln 4 + 2}\right)(16)$

$(\ln 4 + 1)$,which equals $\frac{32(\ln 4 + 1)^2}{\ln 4 + 2}$.

9. $y = \frac{e^x\sqrt{x^2+1}}{(x^2+2)^3}$

To find the slope of the tangent line at $(0, \frac{1}{8})$, we need to evaluate the derivative at $x = 0$.

$$\ln y = \ln\frac{e^x\sqrt{x^2+1}}{(x^2+2)^3}$$

$$\ln y = \ln e^x + \ln\sqrt{x^2+1} - \ln(x^2+2)^3$$

$$\ln y = x\ln e + \frac{1}{2}\ln(x^2+1) - 3\ln(x^2+2)$$

$$\frac{1}{y}\frac{dy}{dx} = 1 + \frac{2x}{2(x^2+1)} - \frac{3(2x)}{x^2+2}$$

$$\frac{dy}{dx} = \frac{1}{y}\left(1 + \frac{x}{x^2+1} - \frac{6x}{x^2+2}\right)$$

$$\frac{dy}{dx} = \frac{e^x\sqrt{x^2+1}}{(x^2+2)^3}\left(1 + \frac{x}{x^2+1} - \frac{6x}{x^2+2}\right)$$

$$m = \frac{e^{(0)}\sqrt{(0)^2+1}}{((0)^2+2)^3}\left(1 + \frac{(0)}{(0)^2+1} - \frac{6(0)}{(0)^2+2}\right)$$

$$m = \frac{1}{8}(1)$$

$$m = \frac{1}{8}$$

10. $f(x) = \left(\frac{x\sin x}{x^2-1}\right)^2$

$$\ln(f(x)) = \ln\left(\frac{x\sin x}{x^2-1}\right)^2$$

$$\ln(f(x)) = 2\ln\frac{x\sin x}{x^2-1}$$

$$\ln(f(x)) = 2\ln(x\sin x) - 2\ln(x^2 - 1)$$

$$\frac{1}{f(x)}f'(x) = \frac{2(\sin x + x\cos x)}{x\sin x} - \frac{2(2x)}{x^2 - 1}$$

$$f'(x) = f(x)\left(\frac{2(\sin x + x\cos x)}{x\sin x} - \frac{4x}{x^2 - 1}\right)$$

$$= \left(\frac{x\sin x}{x^2 - 1}\right)^2\left(\frac{2(\sin x + x\cos x)}{x\sin x} - \frac{4x}{x^2 - 1}\right)$$

11. $y = x^{\cos x}$

$$\ln y = \ln x^{\cos x}$$

$$\ln y = \cos x \ln x$$

$$\frac{1}{y}\frac{dy}{dx} = \sin x \ln x + \frac{\cos x}{x}$$

$$\frac{dy}{dx} = y\left(\sin x \ln x + \frac{\cos x}{x}\right)$$

$$= x^{\cos x}\left(\sin x \ln x + \frac{\cos x}{x}\right)$$

12. $y = x^x$

First use the derivative to find the slope of the tangent line at the point (1, 1)

$$\ln y = \ln x^x$$

$$\ln y = x\ln x$$

$$\frac{1}{y}\frac{dy}{dx} = \ln x + (x)\frac{1}{x}$$

$$\frac{1}{y}\frac{dy}{dx} = \ln x + 1$$

$$\frac{dy}{dx} = y(\ln x + 1)$$

$$= x^x(\ln x + 1)$$

$$m = (1)^{(1)}(\ln(1) + 1)$$

$$= 1(0 + 1) = 1$$

Since we already know a point on the curve, we can use point slope form to write the equation of the tangent line.

$$y - 1 = 1(x - 1)$$

$$y = x - 1 + 1$$

$$y = x$$

13. $s(t) = t^{\frac{1}{t}}, t > 0$

a. $\ln(s(t)) = \frac{1}{t}\ln t$

Differentiate with respect to t:

$$\frac{1}{s(t)} \cdot s'(t) = \frac{\frac{1}{t} \cdot t - \ln t}{t^2}$$

$$= \frac{1 - \ln t}{t^2}.$$

Thus, $v(t) = s(t) \cdot \left(\frac{1 - \ln t}{t^2}\right) = t^{\frac{1}{t}}\left(\frac{1 - \ln t}{t^2}\right).$

Now, $a(t) = v'(t) = s'(t)\left(\frac{1 - \ln t}{t^2}\right) + s(t)$

$$+ s(t)\left(\frac{-\frac{1}{t} \cdot t^2 - (1 - \ln t)(2t)}{t^4}\right)$$

Substituting for $s(t)$ and $s'(t) = v(t)$ gives

$$a(t) = t^{\frac{1}{t}}\left(\frac{1 - \ln t}{t^2}\right)^2 + t^{\frac{1}{t}}\left(\frac{2t\ln t - 3t}{t^4}\right)$$

$$= \frac{t^{\frac{1}{t}}}{t^4}[1 - 2\ln t + (\ln t)^2 + 2t\ln t - 3t]$$

b. Since $t^{\frac{1}{t}}$ and t^2 are always positive, the velocity is zero when

$1 - \ln t = 0$ or when $t = e$.

$$a(e) = \frac{e^{\frac{1}{e}}}{e^4}[1 - 2 + 1 + 2e - 3e]$$

$$= -\frac{e^{\frac{1}{e}}}{e^3}$$

$$= -e^{\frac{1}{e} - 3}$$

14. If $a^b > b^a$, then

$\ln a^b > \ln b^a$

$b\ln a > a\ln b$ Dividing both sides by ab yields

$$\frac{\ln a}{a} > \frac{\ln b}{b}$$

This indicates we should use the function $y = \frac{\ln x}{x}$ to determine whether e^π or π^e is larger.

$$y = \frac{\ln x}{x}$$

$$\frac{dy}{dx} = \frac{1 - \ln x}{x^2}$$

Let $\frac{dy}{dx} = 0$:

$$\frac{1 - \ln x}{x^2} = 0$$

$$1 - \ln x = 0$$

$$x = e$$

Therefore e is a critical number.

x	$x < e$	e	$x > e$
$\frac{dy}{dx}$	+	0	
Graph	Inc.	Local Max	Dec.

The graph attains a local maximum when $x = e$.

Therefore, $\frac{\ln e}{e} > \frac{\ln \pi}{\pi}$, and $e^\pi > \pi^e$.

Using a calculator, $e^\pi \doteq 23.14$ and $\pi^e \doteq 22.46$. This verifies the work above.

Cumulative Review of Calculus, pp. 267–270

1. a. $f(x) = 3x^2 + 4x - 5$

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$
$$= \lim_{h \to 0} \frac{f(2+h) - 15}{h}$$
$$= \lim_{h \to 0} \frac{3(2+h)^2 + 4(2+h) - 5 - 15}{h}$$
$$= \lim_{h \to 0} \frac{12 + 12h + 3h^2 + 8 + 4h - 20}{h}$$
$$= \lim_{h \to 0} \frac{3h^2 + 16h}{h}$$
$$= \lim_{h \to 0} 3h + 16$$
$$= 16$$

b. $f(x) = \frac{2}{x - 1}$

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$
$$= \lim_{h \to 0} \frac{f(2+h) - 2}{h}$$
$$= \lim_{h \to 0} \frac{\frac{2}{2 + h - 1} - 2}{h}$$
$$= \lim_{h \to 0} \frac{\frac{2}{1+h} - \frac{2(1+h)}{1+h}}{h}$$
$$= \lim_{h \to 0} \frac{2 - 2(1+h)}{h(1+h)}$$
$$= \lim_{h \to 0} \frac{-2h}{h(1+h)}$$
$$= \lim_{h \to 0} \frac{-2}{1+h}$$
$$= -2$$

c. $f(x) = \sqrt{x + 3}$

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$
$$= \lim_{h \to 0} \frac{f(6+h) - 3}{h}$$
$$= \lim_{h \to 0} \frac{\sqrt{h+9} - 3}{h}$$
$$= \lim_{h \to 0} \frac{(\sqrt{h+9} - 3)(\sqrt{h+9} + 3)}{h(\sqrt{h+9} + 3)}$$
$$= \lim_{h \to 0} \frac{h + 9 - 9}{h(\sqrt{h+9} + 3)}$$
$$= \lim_{h \to 0} \frac{h}{h(\sqrt{h+9} + 3)}$$
$$= \lim_{h \to 0} \frac{1}{(\sqrt{h+9} + 3)}$$
$$= \frac{1}{6}$$

d. $f(x) = 2^{5x}$

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$
$$= \lim_{h \to 0} \frac{2^{5(1+h)} - 32}{h}$$
$$= \lim_{h \to 0} \frac{2^5 \cdot 2^{5h} - 32}{h}$$
$$= \lim_{h \to 0} \frac{32(2^{5h} - 1)}{h}$$
$$= 32 \lim_{h \to 0} \frac{5(2^{5h} - 1)}{5h}$$
$$= 160 \lim_{h \to 0} \frac{(2^{5h} - 1)}{5h}$$
$$= 160 \ln 2$$

2. a. average velocity $= \frac{\text{change in distance}}{\text{change in time}}$

$$= \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$
$$= \frac{[2(4)^2 + 3(4) + 1] - [(2(1)^2 + 3(1) + 1)]}{4 - 1}$$
$$= \frac{45 - 6}{3}$$
$$= 13 \text{ m/s}$$

b. instantaneous velocity = slope of the tangent

$$m = \lim_{h \to 0} \frac{s(a+h) - s(a)}{h}$$
$$= \lim_{h \to 0} \frac{s(3+h) - s(3)}{h}$$
$$= \lim_{h \to 0} \left[\frac{2(3+h)^2 + 3(3+h) + 1}{h} - \frac{(2(3)^2 + 3(3) + 1)}{h}\right]$$
$$= \lim_{h \to 0} \frac{18 + 12h + 2h^2 + 9 + 3h + 1 - 28}{h}$$

$= \lim_{h\to 0} \frac{15h + 2h^2}{h}$
$= \lim_{h\to 0} (15 + 2h)$
$= 15$ m/s

3. $m = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$

$\lim_{h\to 0} \frac{(4+h)^3 - 64}{h} = \lim_{h\to 0} \frac{f(4+h) - f(4)}{h}$

$(4+h)^3 - 64 = f(4+h) - f(4)$

Therefore, $f(x) = x^3$.

4. a. Average rate of change in distance with respect to time is average velocity, so

average velocity $= \frac{s(t_2) - s(t_1)}{t_2 - t_1}$
$= \frac{s(3) - s(1)}{3 - 1}$
$= \frac{4.9(3)^2 - 4.9(1)}{3 - 1}$
$= 19.6$ m/s

b. Instantaneous rate of change in distance with respect to time $=$ slope of the tangent.

$m = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$
$= \lim_{h\to 0} \frac{f(2+h) - f(2)}{h}$
$= \lim_{h\to 0} \frac{4.9(2+h)^2 - 4.9(2)^2}{h}$
$= \lim_{h\to 0} \frac{19.6 + 19.6h + 4.9h^2 - 19.6}{h}$
$= \lim_{h\to 0} \frac{19.6h + 4.9h^2}{h}$
$= \lim_{h\to 0} 19.6 + 4.9h$
$= 19.6$ m/s

c. First, we need to determine t for the given distance:

$146.9 = 4.9t^2$
$29.98 = t^2$
$5.475 = t$

Now use the slope of the tangent to determine the instantaneous velocity for $t = 5.475$:

$m = \lim_{h\to 0} \frac{f(5.475 + h) - f(5.475)}{h}$
$= \lim_{h\to 0} \frac{4.9(5.475 + h)^2 - 4.9(5.475)^2}{h}$
$= \lim_{h\to 0} \frac{146.9 + 53.655h + 4.9h^2 - 146.9}{h}$
$= \lim_{h\to 0} \frac{53.655h + 4.9h^2}{h}$
$= \lim_{h\to 0} [53.655 + 4.9h]$
$= 53.655$ m/s

5. a. Average rate of population change

$= \frac{p(t_2) - p(t_1)}{t_2 - t_1}$
$= \frac{2(8)^2 + 3(8) + 1 - (2(0) + 3(0) + 1)}{8 - 0}$
$= \frac{128 + 24 + 1 - 1}{8 - 0}$
$= 19$ thousand fish/year

b. Instantaneous rate of population change

$= \lim_{h\to 0} \frac{p(t+h) - p(t)}{h}$
$= \lim_{h\to 0} \frac{p(5+h) - p(5)}{h}$
$= \lim_{h\to 0} \left[\frac{2(5+h)^2 + 3(5+h) + 1}{h} - \frac{(2(5)^2 + 3(5) + 1)}{h}\right]$
$= \lim_{h\to 0} \frac{50 + 20h + 2h^2 + 15 + 3h + 1 - 66}{h}$
$= \lim_{h\to 0} \frac{2h^2 + 23h}{h}$
$= \lim_{h\to 0} 2h + 23$
$= 23$ thousand fish/year

6. a. i. $f(2) = 3$

ii. $\lim_{x\to 2^-} f(x) = 1$

iii. $\lim_{x\to 2^+} f(x) = 3$

iv. $\lim_{x\to 6} f(x) = 2$

b. No, $\lim_{x\to 4} f(x)$ does not exist. In order for the limit to exist, $\lim_{x\to 4^-} f(x)$ and $\lim_{x\to 4^+} f(x)$ must exist and they must be equal. In this case, $\lim_{x\to 4^-} f(x) = \infty$, but $\lim_{x\to 4^+} f(x) = -\infty$, so $\lim_{x\to 4} f(x)$ does not exist.

7. $f(x)$ is discontinuous at $x = 2$. $\lim_{x\to 2^-} f(x) = 5$, but $\lim_{x\to 2^+} f(x) = 3$.

8. a. $\lim_{x\to 0} \frac{2x^2 + 1}{x - 5} = \frac{2(0)^2 + 1}{0 - 5}$
$= -\frac{1}{5}$

b. $\lim_{x\to 3} \frac{x - 3}{\sqrt{x+6} - 3}$
$= \lim_{x\to 3} \frac{(x-3)(\sqrt{x+6} + 3)}{(\sqrt{x+6} - 3)(\sqrt{x+6} + 3)}$

$$= \lim_{x\to 3} \frac{(x-3)(\sqrt{x+6}+3)}{x+6-9}$$
$$= \lim_{x\to 3} \frac{(x-3)(\sqrt{x+6}+3)}{x-3}$$
$$= \lim_{x\to 3} \sqrt{x+6}+3$$
$$= 6$$

c. $\lim_{x\to -3} \frac{\frac{1}{x}+\frac{1}{3}}{x+3}$
$$= \lim_{x\to -3} \frac{\frac{x+3}{3x}}{x+3}$$
$$= \lim_{x\to -3} \frac{x+3}{3x(x+3)}$$
$$= \lim_{x\to -3} \frac{1}{3x}$$
$$= -\frac{1}{9}$$

d. $\lim_{x\to 2} \frac{x^2-4}{x^2-x-2}$
$$= \lim_{x\to 2} \frac{(x+2)(x-2)}{(x+1)(x-2)}$$
$$= \lim_{x\to 2} \frac{x+2}{x+1}$$
$$= \frac{4}{3}$$

e. $\lim_{x\to 2} \frac{x-2}{x^3-8}$
$$= \lim_{x\to 2} \frac{x-2}{(x-2)(x^2+2x+4)}$$
$$= \lim_{x\to 2} \frac{1}{x^2+2x+4}$$
$$= \frac{1}{12}$$

f. $\lim_{x\to 0} \frac{\sqrt{x+4}-\sqrt{4-x}}{x}$
$$= \lim_{x\to 0} \frac{(\sqrt{x+4}-\sqrt{4-x})(\sqrt{x+4}+\sqrt{4-x})}{x(\sqrt{x+4}+\sqrt{4-x})}$$
$$= \lim_{x\to 0} \frac{x+4-(4-x)}{x(\sqrt{x+4}+\sqrt{4-x})}$$
$$= \lim_{x\to 0} \frac{2x}{x(\sqrt{x+4}+\sqrt{4-x})}$$
$$= \lim_{x\to 0} \frac{2}{(\sqrt{x+4}+\sqrt{4-x})}$$
$$= \frac{1}{2}$$

9. a. $f(x) = 3x^2 + x + 1$
$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0} \left[\frac{3(x+h)^2+(x+h)+1}{h} - \frac{(3x^2+x+1)}{h}\right]$$
$$= \lim_{h\to 0} \left[\frac{3x^2+6hx+6h^2+x+h}{h} + \frac{1-3x^2-x-1}{h}\right]$$
$$= \lim_{h\to 0} \frac{6hx+6h^2+h}{h}$$
$$= \lim_{h\to 0} 6x+6h+1$$
$$= 6x+1$$

b. $f(x) = \frac{1}{x}$
$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0} \frac{\frac{1}{x+h}-\frac{1}{x}}{h}$$
$$= \lim_{h\to 0} \frac{x-(x+h)}{h(x)(x+h)}$$
$$= \lim_{h\to 0} \frac{-h}{h(x)(x+h)}$$
$$= \lim_{h\to 0} \frac{-1}{x(x+h)}$$
$$= -\frac{1}{x^2}$$

10. a. To determine the derivative, use the power rule:
$$y = x^3 - 4x^2 + 5x + 2$$
$$\frac{dy}{dx} = 3x^2 - 8x + 5$$

b. To determine the derivative, use the chain rule:
$$y = \sqrt{2x^3+1}$$
$$\frac{dy}{dx} = \frac{1}{2\sqrt{2x^3+1}}(6x^2)$$
$$= \frac{3x^2}{\sqrt{2x^3+1}}$$

c. To determine the derivative, use the quotient rule:
$$y = \frac{2x}{x+3}$$
$$\frac{dy}{dx} = \frac{2(x+3)-2x}{(x+3)^2}$$
$$= \frac{6}{(x+3)^2}$$

d. To determine the derivative, use the product rule:

$$y = (x^2 + 3)^2(4x^5 + 5x + 1)$$

$$\frac{dy}{dx} = 2(x^2 + 3)(2x)(4x^5 + 5x + 1) + (x^2 + 3)^2(20x^4 + 5)$$

$$= 4x(x^2 + 3)(4x^5 + 5x + 1) + (x^2 + 3)^2(20x^4 + 5)$$

e. To determine the derivative, use the quotient rule:

$$y = \frac{(4x^2 + 1)^5}{(3x - 2)^3}$$

$$\frac{dy}{dx} = \frac{5(4x^2 + 1)^4(8x)(3x - 2)^3}{(3x - 2)^6} - \frac{3(3x - 2)^2(3)(4x^2 + 1)^5}{(3x - 2)^6}$$

$$= (4x^2 + 1)^4(3x - 2)^2 \times \frac{40x(3x - 2) - 9(4x^2 + 1)}{(3x - 2)^6}$$

$$= \frac{(4x^2 + 1)^4(120x^2 - 80x - 36x^2 - 9)}{(3x - 2)^4}$$

$$= \frac{(4x^2 + 1)^4(84x^2 - 80x - 9)}{(3x - 2)^4}$$

f. $y = [x^2 + (2x + 1)^3]^5$

Use the chain rule

$$\frac{dy}{dx} = 5[x^2 + (2x + 1)^3]^4[2x + 6(2x + 1)^2]$$

11. To determine the equation of the tangent line, we need to determine its slope at the point $(1, 2)$. To do this, determine the derivative of y and evaluate for $x = 1$:

$$y = \frac{18}{(x + 2)^2}$$

$$= 18(x + 2)^{-2}$$

$$\frac{dy}{dx} = -36(x + 2)^{-3}$$

$$= \frac{-36}{(x + 2)^3}$$

$$m = \frac{-36}{(x + 2)^3}$$

$$= \frac{-36}{27} = \frac{-4}{3}$$

Since we have a given point and we know the slope, use point-slope form to write the equation of the tangent line:

$$y - 2 = \frac{-4}{3}(x - 1)$$

$$3y - 6 = -4x + 4$$

$$4x + 3y - 10 = 0$$

12. The intersection point of the two curves occurs when

$$x^2 + 9x + 9 = 3x$$

$$x^2 + 6x + 9 = 0$$

$$(x + 3)^2 = 0$$

$$x = -3.$$

At a point x, the slope of the line tangent to the curve $y = x^2 + 9x + 9$ is given by

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 + 9x + 9)$$

$$= 2x + 9.$$

At $x = -3$, this slope is $2(-3) + 9 = 3$.

13. a. $p'(t) = \frac{d}{dt}(2t^2 + 6t + 1100)$

$$= 4t + 6$$

b. 1990 is 10 years after 1980, so the rate of change of population in 1990 corresponds to the value

$p'(10) = 4(10) + 6$

$= 46$ people per year.

c. The rate of change of the population will be 110 people per year when

$$4t + 6 = 110$$

$$t = 26.$$

This corresponds to 26 years after 1980, which is the year 2006.

14. a. $f'(x) = \frac{d}{dx}(x^5 - 5x^3 + x + 12)$

$$= 5x^4 - 15x^2 + 1$$

$$f''(x) = \frac{d}{dx}(5x^4 - 15x^2 + 1)$$

$$= 20x^3 - 30x$$

b. $f(x)$ can be rewritten as $f(x) = -2x^{-2}$

$$f'(x) = \frac{d}{dx}(-2x^{-2})$$

$$= 4x^{-3}$$

$$= \frac{4}{x^3}$$

$$f''(x) = \frac{d}{dx}(4x^{-3})$$

$$= -12x^{-4}$$

$$= -\frac{12}{x^4}$$

c. $f(x)$ can be rewritten as $f(x) = 4x^{-\frac{1}{2}}$

$$f'(x) = \frac{d}{dx}(4x^{-\frac{1}{2}})$$

$$= -2x^{-\frac{3}{2}}$$

$$= -\frac{2}{\sqrt{x^3}}$$

$$f''(x) = \frac{d}{dx}(-2x^{-\frac{3}{2}})$$
$$= 3x^{-\frac{5}{2}}$$
$$= \frac{3}{\sqrt{x^5}}$$

d. $f(x)$ can be rewritten as $f(x) = x^4 - x^{-4}$

$$f'(x) = \frac{d}{dx}(x^4 - x^{-4})$$
$$= 4x^3 + 4x^{-5}$$
$$= 4x^3 + \frac{4}{x^5}$$
$$f''(x) = \frac{d}{dx}(4x^3 + 4x^{-5})$$
$$= 12x^2 - 20x^{-6}$$
$$= 12x^2 - \frac{20}{x^6}$$

15. Extreme values of a function on an interval will only occur at the endpoints of the interval or at a critical point of the function.

a. $f'(x) = \frac{d}{dx}(1 + (x + 3)^2)$
$$= 2(x + 3)$$

The only place where $f'(x) = 0$ is at $x = -3$, but that point is outside of the interval in question. The extreme values therefore occur at the endpoints of the interval:

$f(-2) = 1 + (-2 + 3)^2 = 2$
$f(6) = 1 + (6 + 3)^2 = 82$

The maximum value is 82, and the minimum value is 6

b. $f(x)$ can be rewritten as $f(x) = x + x^{-\frac{1}{2}}$

$$f'(x) = \frac{d}{dx}(x + x^{-\frac{1}{2}})$$
$$= 1 + -\frac{1}{2}x^{-\frac{3}{2}}$$
$$= 1 - \frac{1}{2\sqrt{x^3}}$$

On this interval, $x \geq 1$, so the fraction on the right is always less than or equal to $\frac{1}{2}$. This means that $f'(x) > 0$ on this interval and so the extreme values occur at the endpoints.

$$f(1) = 1 + \frac{1}{\sqrt{1}} = 2$$
$$f(9) = 9 + \frac{1}{\sqrt{9}} = 9\frac{1}{3}$$

The maximum value is $9\frac{1}{3}$, and the minimum value is 2.

c. $f'(x) = \frac{d}{dx}\left(\frac{e^x}{1 + e^x}\right)$
$$= \frac{(1 + e^x)(e^x) - (e^x)(e^x)}{(1 + e^x)^2}$$
$$= \frac{e^x}{(1 + e^x)^2}$$

Since e^x is never equal to zero, $f'(x)$ is never zero, and so the extreme values occur at the endpoints of the interval.

$$f(0) = \frac{e^0}{1 + e^0} = \frac{1}{2}$$
$$f(4) = \frac{e^4}{1 + e^4}$$

The maximum value is $\frac{e^4}{1 + e^4}$, and the minimum value is $\frac{1}{2}$.

d. $f'(x) = \frac{d}{dx}(2\sin(4x) + 3)$
$$= 8\cos(4x)$$

Cosine is 0 when its argument is a multiple of $\frac{\pi}{2}$ or $\frac{3\pi}{2}$.

$$4x = \frac{\pi}{2} + 2k\pi \text{ or } 4x = \frac{3\pi}{2} + 2k\pi$$
$$x = \frac{\pi}{8} + \frac{\pi}{2}k \qquad x = \frac{3\pi}{8} + \frac{\pi}{2}k$$

Since $x \in [0, \pi]$, $x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$.

Also test the function at the endpoints of the interval.

$$f(0) = 2\sin 0 + 3 = 3$$
$$f\left(\frac{\pi}{8}\right) = 2\sin\frac{\pi}{2} + 3 = 5$$
$$f\left(\frac{3\pi}{8}\right) = 2\sin\frac{3\pi}{2} + 3 = 1$$
$$f\left(\frac{5\pi}{8}\right) = 2\sin\frac{5\pi}{2} + 3 = 5$$
$$f\left(\frac{7\pi}{8}\right) = 2\sin\frac{7\pi}{2} + 3 = 1$$
$$f(\pi) = 2\sin(4\pi) + 3 = 3$$

The maximum value is 5, and the minimum value is 1.

16. a. The velocity of the particle is given by

$$v(t) = s'(t)$$
$$= \frac{d}{dt}(3t^3 - 40.5t^2 + 162t)$$
$$= 9t^2 - 81t + 162.$$

The acceleration is

$$a(t) = v'(t)$$
$$= \frac{d}{dt}(9t^2 - 81t + 162)$$
$$= 18t - 81$$

b. The object is stationary when $v(t) = 0$:

$$9t^2 - 81t + 162 = 0$$
$$9(t - 6)(t - 3) = 0$$
$$t = 6 \text{ or } t = 3$$

The object is advancing when $v(t) > 0$ and retreating when $v(t) < 0$. Since $v(t)$ is the product of two linear factors, its sign can be determined using the signs of the factors:

t-values	*t* − 3	*t* − 6	*v*(*t*)	Object
$0 < t < 3$	< 0	< 0	> 0	Advancing
$3 < t < 6$	> 0	< 0	< 0	Retreating
$6 < t < 8$	> 0	> 0	> 0	Advancing

c. The velocity of the object is unchanging when the acceleration is 0; that is, when

$$a(t) = 18t - 81 = 0$$
$$t = 4.5$$

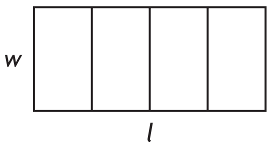
d. The object is decelerating when $a(t) < 0$, which occurs when

$$18t - 81 < 0$$
$$0 \le t < 4.5$$

e. The object is accelerating when $a(t) > 0$, which occurs when

$$18t - 81 > 0$$
$$4.5 < t \le 8$$

17.



Let the length and width of the field be l and w, as shown. The total amount of fencing used is then $2l + 5w$. Since there is 750 m of fencing available, this gives

$$2l + 5w = 750$$
$$l = 375 - \frac{5}{2}w$$

The total area of the pens is

$$A = lw$$
$$= 375w - \frac{5}{2}w^2$$

The maximum value of this area can be found by expressing A as a function of w and examining its derivative to determine critical points.

$A(w) = 375w - \frac{5}{2}w^2$, which is defined for $0 \le w$ and $0 \le l$. Since $l = 375 - \frac{5}{2}w$, $0 \le l$ gives the restriction $w \le 150$. The maximum area is therefore the maximum value of the function $A(w)$ on the interval $0 \le w \le 150$.

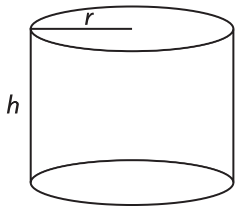
$$A'(w) = \frac{d}{dw}\left(375w - \frac{5}{2}w^2\right)$$
$$= 375 - 5w$$

Setting $A'(w) = 0$ shows that $w = 75$ is the only critical point of the function. The only values of interest are therefore:

$$A(0) = 375(0) - \frac{5}{2}(0)^2 = 0$$
$$A(75) = 375(75) - \frac{5}{2}(75)^2 = 14\,062.5$$
$$A(150) = 375(150) - \frac{5}{2}(150)^2 = 0$$

The maximum area is 14 062.5 m^2

18.



Let the height and radius of the can be h and r, as shown. The total volume of the can is then $\pi r^2 h$. The volume of the can is also give at 500 mL, so

$$\pi r^2 h = 500$$
$$h = \frac{500}{\pi r^2}$$

The total surface area of the can is

$$A = 2\pi rh + 2\pi r^2$$
$$= \frac{1000}{r} + 2\pi r^2$$

The minimum value of this surface area can be found by expressing A as a function of r and examining its derivative to determine critical points.

$A(r) = \frac{1000}{r} + 2\pi r^2$, which is defined for $0 < r$ and $0 < h$. Since $h = \frac{500}{\pi r^2}$, $0 < h$ gives no additional restriction on r. The maximum area is therefore the maximum value of the function $A(r)$ on the interval $0 < r$.

$$A'(r) = \frac{d}{dr}\left(\frac{1000}{r} + 2\pi r^2\right)$$
$$= -\frac{1000}{r^2} + 4\pi r$$

The critical points of $A(r)$ can be found by setting $A'(r) = 0$:

$$-\frac{1000}{r^2} + 4\pi r = 0$$

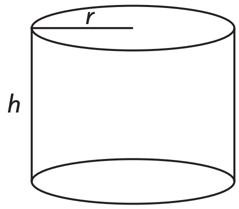
$$4\pi r^3 = 1000$$

$$r = \sqrt[3]{\frac{1000}{4\pi}} \doteq 4.3 \text{ cm}$$

So $r = 4.3$ cm is the only critical point of the function. This gives the value

$$h = \frac{500}{\pi(4.3)^2} \doteq 8.6 \text{ cm.}$$

19.



Let the radius be r and the height h.
Minimize the cost:

$$C = 2\pi r^2(0.005) + 2\pi rh(0.0025)$$

$$V = \pi r^2 h = 4000$$

$$h = \frac{4000}{\pi r^2}$$

$$C(r) = 2\pi r^2(0.005) + 2\pi r\left(\frac{4000}{\pi r^2}\right)(0.0025)$$

$$= 0.01\pi r^2 + \frac{20}{r}, 1 \leq r \leq 36$$

$$C'(r) = 0.02\pi r - \frac{20}{r^2}.$$

For a maximum or minimum value, let $C'(r) = 0$.

$$0.02\pi r^2 - \frac{20}{r^2} = 0$$

$$r^3 = \frac{20}{0.02\pi}$$

$$r \doteq 6.8$$

Using the max min algorithm:
$C(1) = 20.03$, $C(6.8) = 4.39$, $C(36) = 41.27$.
The dimensions for the cheapest container are a radius of 6.8 cm and a height of 27.5 cm.

20. a. Let the length, width, and depth be l, w, and d, respectively. Then, the given information is that $l = x$, $w = x$, and
$l + w + d = 140$. Substituting gives

$$2x + d = 140$$

$$d = 140 - 2x$$

b. The volume of the box is $V = lwh$. Substituting in the values from part **a**. gives

$$V = (x)(x)(140 - 2x)$$

$$= 140x^2 - 2x^3$$

In order for the dimensions of the box to make sense, the inequalities $l \geq 0$, $w \geq 0$, and $h \geq 0$ must be satisfied. The first two give $x \geq 0$, the third requires $x \leq 70$. The maximum volume is therefore the maximum value of $V(x) = 140x^2 - 2x^3$ on the interval $0 \leq x \leq 70$, which can be found by determining the critical points of the derivative $V'(x)$.

$$V'(x) = \frac{d}{dx}(140x^2 - 2x^3)$$

$$= 280x - 6x^2$$

$$= 2x(140 - 3x)$$

Setting $V'(x) = 0$ shows that $x = 0$ and $x = \frac{140}{3} \doteq 46.7$ are the critical points of the function.
The maximum value therefore occurs at one of these points or at one of the endpoints of the interval:

$$V(0) = 140(0)^2 - 2(0)^3 = 0$$

$$V(46.7) = 140(46.7)^2 - 2(46.7)^3 = 101\,629.5$$

$$V(0) = 140(70)^2 - 2(70)^3 = 0$$

So the maximum volume is 101 629.5 cm^3, from a box with length and width 46.7 cm and depth $140 - 2(46.7) = 46.6$ cm.

21. The revenue function is

$$R(x) = x(50 - x^2)$$

$= 50x - x^3$. Its maximum for $x \geq 0$ can be found by examining its derivative to determine critical points.

$$R'(x) = \frac{d}{dx}(50x - x^3)$$

$$= 50 - 3x^2$$

The critical points can be found by setting $R'(x) = 0$:

$$50 - 3x^2 = 0$$

$$x = \pm\sqrt{\frac{50}{3}} \doteq \pm 4.1$$

Only the positive root is of interest since the number of MP3 players sold must be positive. The number must also be an integer, so both $x = 4$ and $x = 5$ must be tested to see which is larger.

$$R(4) = 50(4) - 4^3 = 136$$

$$R(4) = 50(5) - 5^3 = 125$$

So the maximum possible revenue is \$136, coming from a sale of 4 MP3 players.

22. Let x be the fare, and $p(x)$ be the number of passengers per year. The given information shows that p is a linear function of x such that an increase of 10 in x results in a decrease of 1000 in p. This means that the slope of the line described by $p(x)$ is $\frac{-1000}{10} = -100$. Using the initial point given,

$$p(x) = -100(x - 50) + 10\,000$$

$$= -100x + 15\,000$$

The revenue function can now be written:

$R(x) = xp(x)$
$= x(-100x + 15\,000)$
$= 15\,000x - 100x^2$

Its maximum for $x \geq 0$ can be found by examining its derivative to determine critical points.

$$R'(x) = \frac{d}{dx}(15\,000x - 100x^2)$$
$$= 15\,000 - 200x$$

Setting $R'(x) = 0$ shows that $x = 75$ is the only critical point of the function. The problem states that only $10 increases in fare are possible, however, so the two nearest must be tried to determine the maximum possible revenue:

$R(70) = 15\,000(70) - 100(70)^2 = 560\,000$
$R(80) = 15\,000(80) - 100(80)^2 = 560\,000$

So the maximum possible revenue is $560 000, which can be achieved by a fare of either $70 or $80.

23. Let the number of $30 price reductions be n. The resulting number of tourists will be $80 + n$ where $0 \leq n \leq 70$. The price per tourist will be $5000 - 30n$ dollars. The revenue to the travel agency will be $(5000 - 30n)(80 + n)$ dollars. The cost to the agency will be $250\,000 + 300(80 + n)$ dollars.

Profit = Revenue − Cost

$P(n) = (5000 - 30n)(80 + n)$
$- 250\,000 - 300(80 + n), 0 \leq n \leq 70$
$P'(n) = -30(80 + n) + (5000 - 30n)(1) - 300$
$= 2300 - 60n$

$P'(n) = 0$ when $n = 38\frac{1}{3}$

Since n must be an integer, we now evaluate $P(n)$ for $n = 0, 38, 39$, and 70. (Since $P(n)$ is a quadratic function whose graph opens downward with vertex at $38\frac{1}{3}$, we know $P(38) > P(39)$.)

$P(0) = 126\,000$
$P(38) = (3860)(118) - 250\,000 - 300(118)$
$= 170\,080$
$P(39) = (3830)(119) - 250\,000 - 300(119)$
$= 170\,070$
$P(70) = (2900)(150) - 250\,000 - 300(150)$
$= 140\,000$

The price per person should be lowered by $1140 (38 decrements of $30) to realize a maximum profit of $170 080.

24. a. $\frac{dy}{dx} = \frac{d}{dx}(-5x^2 + 20x + 2)$
$= -10x + 20$

Setting $\frac{dy}{dx} = 0$ shows that $x = 2$ is the only critical number of the function.

x	$x < 2$	$x = 2$	$x > 2$
y′	+	0	−
Graph	Inc.	Local Max	Dec.

b. $\frac{dy}{dx} = \frac{d}{dx}(6x^2 + 16x - 40)$
$= 12x + 16$

Setting $\frac{dy}{dx} = 0$ shows that $x = -\frac{4}{3}$ is the only critical number of the function.

x	$x < -\frac{4}{3}$	$x = -\frac{4}{3}$	$x > -\frac{4}{3}$
y′	−	0	+
Graph	Dec.	Local Min	Inc.

c. $\frac{dy}{dx} = \frac{d}{dx}(2x^3 - 24x)$
$= 6x^2 - 24$

The critical numbers are found by setting $\frac{dy}{dx} = 0$:

$6x^2 - 24 = 0$
$6x^2 = 24$
$x = \pm 2$

x	$x < -2$	$x = -2$	$-2 < x < 2$	$x = 2$	$x > 2$
y′	+	0	−	0	+
Graph	Inc.	Local Max	Dec.	Local Min	Inc.

d. $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{x - 2}\right)$
$$= \frac{(x - 2)(1) - x(1)}{(x - 2)^2}$$
$$= \frac{-2}{(x - 2)^2}$$

This derivative is never equal to zero, so the function has no critical numbers. Since the numerator is always negative and the denominator is never negative, the derivative is always negative. This means that the function is decreasing everywhere it is defined, that is, $x \neq 2$.

25. a. This function is discontinuous when
$x^2 - 9 = 0$
$x = \pm 3$. The numerator is non-zero at these points, so these are the equations of the vertical asymptotes.

To check for a horizontal asymptote:

$$\lim_{x\to\infty} \frac{8}{x^2-9} = \lim_{x\to\infty} \frac{8}{x^2\left(1-\frac{9}{x^2}\right)}$$
$$= \frac{\lim_{x\to\infty}(8)}{\lim_{x\to\infty} x^2\left(1-\frac{9}{x^2}\right)}$$
$$= \frac{\lim_{x\to\infty}(8)}{\lim_{x\to\infty}(x)^2 \times \lim_{x\to\infty}\left(1-\frac{9}{x^2}\right)}$$
$$= \lim_{x\to\infty} \frac{1}{x^2} \times \frac{8}{1-0}$$
$$= 0$$

Similarly, $\lim_{x\to-\infty} \frac{8}{x^2-9} = 0$, so $y = 0$ is a horizontal asymptote of the function.

There is no oblique asymptote because the degree of the numerator does not exceed the degree of the denominator by 1.

Local extrema can be found by examining the derivative to determine critical points:

$$y' = \frac{(x^2-9)(0)-(8)(2x)}{(x^2-9)^2}$$
$$= \frac{-16x}{(x^2-9)^2}$$

Setting $y' = 0$ shows that $x = 0$ is the only critical point of the function.

x	$x < 0$	$x = 0$	$x > 0$
y'	+	0	+
Graph	Inc.	Local Max	Dec.

So $(0, -\frac{8}{9})$ is a local maximum.

b. This function is discontinuous when
$x^2 - 1 = 0$

$x = \pm 1$. The numerator is non-zero at these points, so these are the equations of the vertical asymptotes.

To check for a horizontal asymptote:

$$\lim_{x\to\infty} \frac{4x^3}{x^2-1} = \lim_{x\to\infty} \frac{x^3(4)}{x^2\left(1-\frac{1}{x^2}\right)}$$
$$= \lim_{x\to\infty} \frac{x(4)}{1-\frac{1}{x^2}}$$
$$= \frac{\lim_{x\to\infty}(x(4))}{\lim_{x\to\infty}\left(1-\frac{1}{x^2}\right)}$$
$$= \frac{\lim_{x\to\infty}(x) \times \lim_{x\to\infty}(4)}{\lim_{x\to\infty}\left(1-\frac{1}{x^2}\right)}$$
$$= \lim_{x\to\infty}(x) \times \frac{4}{1-0}$$
$$= \infty$$

Similarly, $\lim_{x\to-\infty} \frac{4x^3}{x^2-1} = \lim_{x\to-\infty}(x) = -\infty$, so this function has no horizontal asymptote.

To check for an oblique asymptote:

$$\begin{array}{r} 4x \\ x^2-1\overline{)4x^3+0x^2+0x+0} \\ \underline{4x^3+0x^2-4x} \\ 0\quad+0\quad+4x+0 \end{array}$$

So y can be written in the form $y = 4x + \frac{4x}{x^2-1}$. Since

$$\lim_{x\to\infty} \frac{4x}{x^2-1} = \lim_{x\to\infty} \frac{x(4)}{x^2\left(1-\frac{1}{x^2}\right)}$$
$$= \lim_{x\to\infty} \frac{4}{x\left(1-\frac{1}{x^2}\right)}$$
$$= \frac{\lim_{x\to\infty}(4)}{\lim_{x\to\infty}\left(x\left(1-\frac{1}{x^2}\right)\right)}$$
$$= \frac{\lim_{x\to\infty}(4)}{\lim_{x\to\infty}(x) \times \lim_{x\to\infty}\left(1-\frac{1}{x^2}\right)}$$
$$= \lim\left(\frac{1}{x}\right) \times \frac{4}{1-0}$$
$$= 0,$$

and similarly $\lim_{x\to-\infty} \frac{4x}{x^2-1} = 0$, the line $y = 4x$ is an asymptote to the function y.

Local extrema can be found by examining the derivative to determine critical points:

$$y' = \frac{(x^2-1)(12x^2)-(4x^3)(2x)}{(x^2-1)^2}$$
$$= \frac{12x^4-12x^2-8x^4}{(x^2-1)^2}$$
$$= \frac{4x^4-12x^2}{(x^2-1)^2}$$

Setting $y' = 0$:

$4x^4 - 12x^2 = 0$

$x^2(x^2 - 3) = 0$

so $x = 0, x = \pm\sqrt{3}$ are the critical points of the function

x	$x < -\sqrt{3}$	$x = -\sqrt{3}$	$-\sqrt{3} < x < 0$	$x = 0$
y'	+	0	−	0
Graph	Inc.	Local Max	Dec.	Horiz.

x	$0 < x < \sqrt{3}$	$x = \sqrt{3}$	$x > \sqrt{3}$
y'	−	0	−
Graph	Dec.	Local Min	Inc.

$(-\sqrt{3}, -6\sqrt{3})$ is a local maximum, $(\sqrt{3}, 6\sqrt{3})$ is a local minimum, and $(0, 0)$ is neither.

26. a. This function is continuous everywhere, so it has no vertical asymptotes. To check for a horizontal asymptote:

$$\lim_{x\to\infty}(4x^3 + 6x^2 - 24x - 2)$$
$$= \lim_{x\to\infty} x^3\left(4 + \frac{6}{x} - \frac{24}{x^2} - \frac{2}{x^3}\right)$$
$$= \lim_{x\to\infty}(x^3) \times \lim_{x\to\infty}\left(4 + \frac{6}{x} - \frac{24}{x^2} - \frac{2}{x^3}\right)$$
$$= \lim_{x\to\infty}(x^3) \times (4 + 0 - 0 - 0)$$
$$= \infty$$

Similarly,
$\lim_{x\to-\infty}(4x^3 + 6x^2 - 24x - 2) = \lim_{x\to-\infty}(x^3) = -\infty$,
so this function has no horizontal asymptote.

The y-intercept can be found by letting $x = 0$, which gives $y = 4(0)^3 + 6(0)^2 - 24(0) - 2$
$= -2$

The derivative is of the function is

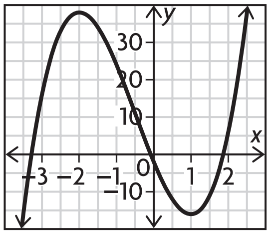
$$y' = \frac{d}{dx}(4x^3 + 6x^2 - 24x - 2)$$
$$= 12x^2 + 12x - 24$$
$= 12(x + 2)(x - 1)$, and the second derivative is

$$y'' = \frac{d}{dx}(12x^2 + 12x - 24)$$
$$= 24x + 12$$

Letting $f'(x) = 0$ shows that $x = -2$ and $x = 1$ are critical points of the function. Letting $y'' = 0$ shows that $x = -\frac{1}{2}$ is an inflection point of the function.

x	$x < -2$	$x = -2$	$-2 < x$	$x = -\frac{1}{2}$
y'	+	0	−	−
Graph	Inc.	Local Max	Dec.	Dec.
y''	−	−	−	0
Concavity	Down	Down	Down	Infl.

x	$-\frac{1}{2} < x < 1$	$x = 1$	$x > 1$
y'	−	0	+
Graph	Dec.	Local Min	Inc.
y''	+	+	+
Concavity	Up	Up	Up



$y = 4x^3 + 6x^2 - 24x - 2$

b. This function is discontinuous when

$$x^2 - 4 = 0$$
$$(x + 2)(x - 2) = 0$$

$x = 2$ or $x = -2$. The numerator is non-zero at these points, so the function has vertical asymptotes at both of them. The behaviour of the function near these asymptotes is:

x-values	$3x$	$x + 2$	$x - 2$	y	$\lim_{x\to\infty} y$
$x \to -2^-$	< 0	< 0	< 0	< 0	$-\infty$
$x \to -2^+$	< 0	> 0	< 0	> 0	$+\infty$
$x \to 2^-$	> 0	> 0	< 0	< 0	$-\infty$
$x \to 2^+$	> 0	> 0	> 0	> 0	$+\infty$

To check for a horizontal asymptote:

$$\lim_{x\to\infty}\frac{3x}{x^2 - 4} = \lim_{x\to\infty}\frac{x(3)}{x^2\left(1 - \frac{4}{x^2}\right)}$$
$$= \lim_{x\to\infty}\frac{3}{x\left(1 - \frac{4}{x^2}\right)}$$
$$= \frac{\lim_{x\to\infty}(3)}{\lim_{x\to\infty}\left(x\left(1 - \frac{4}{x^2}\right)\right)}$$
$$= \frac{\lim_{x\to\infty}(3)}{\lim_{x\to\infty}(x) \times \lim_{x\to\infty}\left(1 - \frac{4}{x^2}\right)}$$
$$= \lim_{x\to\infty}\frac{1}{x} \times \frac{3}{1 - 0}$$
$$= 0$$

Similarly, $\lim_{x\to\infty}\frac{3x}{x^2 - 4} = 0$, so $y = 0$ is a horizontal asymptote of the function.

This function has $y = 0$ when $x = 0$, so the origin is both the x- and y-intercept.
The derivative is

$$y' = \frac{(x^2 - 4)(3) - (3x)(2x)}{(x^2 - 4)^2}$$

$$= \frac{-3x^2 - 12}{(x^2 - 4)^2}$$, and the second derivative is

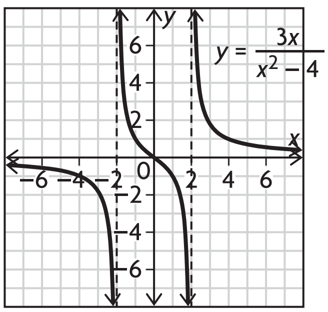
$$y'' = \frac{(x^2 - 4)^2(-6x)}{(x^2 - 4)^4} - \frac{(-3x^2 - 12)(2(x^2 - 4)(2x))}{(x^2 - 4)^4}$$

$$= \frac{-6x^3 + 24x + 12x^3 + 48x}{(x^2 - 4)^3}$$

$$= \frac{6x^3 + 72x}{(x^2 - 4)^3}$$

The critical points of the function can be found by letting $y' = 0$, so
$-3x^2 - 12 = 0$
$x^2 + 4 = 0$. This has no real solutions, so the function y has no critical points.
The inflection points can be found by letting $y'' = 0$, so
$6x^3 + 72x = 0$
$6x(x^2 + 12) = 0$
The only real solution to this equation is $x = 0$, so that is the only possible inflection point.

x	$x < -2$	$-2 < x < 0$	$x = 0$	$0 < x < 2$	$x > 2$
y'	−	−	−	−	−
Graph	Dec.	Dec.	Dec.	Dec.	Dec.
y''	−	+	0	−	+
Concavity	Down	Up	Infl.	Down	Up



27. a. $f'(x) = \frac{d}{dx}((-4)e^{5x+1})$
$= (-4)e^{5x+1} \times \frac{d}{dx}(5x + 1)$
$= (-20)e^{5x+1}$

b. $f'(x) = \frac{d}{dx}(xe^{3x})$
$= xe^{3x} \times \frac{d}{dx}(3x) + (1)e^{3x}$
$= e^{3x}(3x + 1)$

c. $y' = \frac{d}{dx}(6^{3x-8})$
$= (\ln 6)6^{3x-8} \times \frac{d}{dx}(3x - 8)$
$= (3\ln 6)6^{3x-8}$

d. $y' = \frac{d}{dx}(e^{\sin x})$
$= e^{\sin x} \times \frac{d}{dx}(\sin x)$
$= (\cos x)e^{\sin x}$

28. The slope of the tangent line at $x = 1$ can be found by evaluating the derivative $\frac{dy}{dx}$ for $x = 1$:

$\frac{dy}{dx} = \frac{d}{dx}(e^{2x-1})$
$= e^{2x-1} \times \frac{d}{dx}(2x - 1)$
$= 2e^{2x-1}$

Substituting $x = 1$ shows that the slope is $2e$. The value of the original function at $x = 1$ is e, so the equation of the tangent line at $x = 1$ is $y = 2e(x - 1) + e$.

29. a. The maximum of the function modelling the number of bacteria infected can be found by examining its derivative.

$N'(t) = \frac{d}{dt}((15t)e^{-\frac{t}{5}})$
$= 15te^{-\frac{t}{5}} \times \frac{d}{dt}\left(-\frac{t}{5}\right) + (15)e^{-\frac{t}{5}}$
$= e^{-\frac{t}{5}}(15 - 3t)$

Setting $N'(t) = 0$ shows that $t = 5$ is the only critical point of the function (since the exponential function is never zero). The maximum number of infected bacteria therefore occurs after 5 days.

b. $N(5) = (15(5))e^{-\frac{5}{5}}$
$= 27$ bacteria

30. a. $\frac{dy}{dx} = \frac{d}{dx}(2 \sin x - 3 \cos 5x)$
$= 2 \cos x - 3(-\sin 5x) \times \frac{d}{dx}(5x)$
$= 2 \cos x + 15 \sin 5x$

b. $\frac{dy}{dx} = \frac{d}{dx}(\sin 2x + 1)^4$

$= 4(\sin 2x + 1)^3 \times \frac{d}{dx}(\sin 2x + 1)$

$= 4(\sin 2x + 1)^3 \times (\cos 2x) \times \frac{d}{dx}(2x)$

$= 8 \cos 2x(\sin 2x + 1)^3$

c. y can be rewritten as $y = (x^2 + \sin 3x)^{\frac{1}{2}}$. Then,

$\frac{dy}{dx} = \frac{d}{dx}(x^2 + \sin 3x)^{\frac{1}{2}}$

$= \frac{1}{2}(x^2 + \sin 3x)^{-\frac{1}{2}} \times \frac{d}{dx}(x^2 + \sin 3x)$

$= \frac{1}{2}(x^2 + \sin 3x)^{-\frac{1}{2}}$

$\times \left(2x + \cos 3x \times \frac{d}{dx}(3x)\right)$

$= \frac{2x + 3\cos 3x}{2\sqrt{x^2 + \sin 3x}}$

d. $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\sin x}{\cos x + 2}\right)$

$= \frac{(\cos x + 2)(\cos x) - (\sin x)(-\sin x)}{(\cos x + 2)^2}$

$= \frac{\cos^2 x + \sin^2 x + 2\cos x}{(\cos x + 2)^2}$

$= \frac{1 + 2\cos x}{(\cos x + 2)^2}$

e. $\frac{dy}{dx} = \frac{d}{dx}(\tan x^2 - \tan^2 x)$

$= \frac{d}{dx}\sec^2 x^2 \times \frac{d}{dx}(x^2)$

$- 2\tan x \times \frac{d}{dx}(\tan x)$

$= 2x \sec^2 x^2 - 2\tan x \sec^2 x$

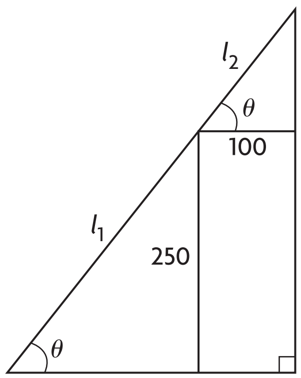
f. $\frac{dy}{dx} = \frac{d}{dx}(\sin(\cos x^2))$

$= \cos(\cos x^2) \times \frac{d}{dx}(\cos x^2)$

$= \cos(\cos x^2) \times (-\sin x^2) \times \frac{d}{dx}(x^2)$

$= -2x \sin x^2 \cos(\cos x^2)$

31.



As shown in the diagram, let θ be the angle between the ladder and the ground, and let the total length of the ladder be $l = l_1 + l_2$, where l_1 is the length from the ground to the top corner of the shed and $l_2$is the length from the corner of the shed to the wall.

$\sin\theta = \frac{250}{l_1} \qquad \cos\theta = \frac{100}{l_2}$

$l_1 = 250\csc\theta \qquad l_2 = 100\sec\theta$

$l = 250\csc\theta + 100\sec\theta$

$\frac{dl}{d\theta} = -250\csc\theta\cot\theta + 100\sec\theta\tan\theta$

$= -\frac{250\cos\theta}{\sin^2\theta} + \frac{100\sin\theta}{\cos^2\theta}$

To determine the minimum, solve $\frac{dl}{d\theta} = 0$.

$\frac{250\cos\theta}{\sin^2\theta} = \frac{100\sin\theta}{\cos^2\theta}$

$250\cos^3\theta = 100\sin^3\theta$

$2.5 = \tan^3\theta$

$\tan\theta = \sqrt[3]{2.5}$

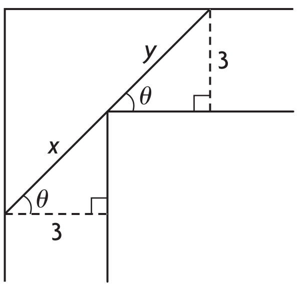
$\theta \doteq 0.94$

At $\theta = 0.94$, $l = 250\csc 0.94 + 100\sec 0.94$

$\doteq 479$ cm

The shortest ladder is about 4.8 m long.

32. The longest rod that can fit around the corner is determined by the minimum value of $x + y$. So, determine the minimum value of $l = x + y$.



From the diagram, $\sin\theta = \frac{3}{y}$ and $\cos\theta = \frac{3}{x}$. So,

$l = \frac{3}{\cos\theta} + \frac{3}{\sin\theta}$, for $0 \le \theta \le \frac{\pi}{2}$.

$$\frac{dl}{d\theta} = \frac{3\sin\theta}{\cos^2\theta} - \frac{3\cos\theta}{\sin^2\theta}$$
$$= \frac{3\sin^3\theta - 3\cos^3\theta}{\cos^2\theta\sin^2\theta}$$

Solving $\frac{dl}{d\theta} = 0$ yields:

$$3\sin^3\theta - 3\cos^3\theta = 0$$
$$\tan^3\theta = 1$$
$$\tan\theta = 1$$
$$\theta = \frac{\pi}{4}$$

So $l = \frac{3}{\cos\frac{\pi}{4}} + \frac{3}{\sin\frac{\pi}{4}}$
$$= 3\sqrt{2} + 3\sqrt{2}$$
$$= 6\sqrt{2}$$

When $\theta = 0$ or $\theta = \frac{\pi}{2}$, the longest possible rod would have a length of 3 m. Therefore the longest rod that can be carried horizontally around the corner is one of length $6\sqrt{2}$, or about 8.5 m.

Cumulative Review of Vectors pp. 557–560

1. a. The angle, θ, between the two vectors is found from the equation $\cos(\theta) = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$.

$$\vec{a} \cdot \vec{b} = (2, -1, -2) \cdot (3, -4, 12) = 2(3) - 1(-4) - 2(12) = -14$$

$$|\vec{a}| = \sqrt{2^2 + (-1)^2 + (-2)^2} = 3$$

$$|\vec{b}| = \sqrt{3^2 + (-4)^2 + 12^2} = 13$$

So $\theta = \cos^{-1}\left(\frac{-14}{3 \times 13}\right) \doteq 111.0°$

b. The scalar projection of $\vec{a}$ on $\vec{b}$ is equal to $|\vec{a}|\cos(\theta)$, where θ is the angle between the two vectors. So from the above work, $\cos(\theta) = \frac{-14}{3 \times 13}$ and $|\vec{a}| = 3$, so the scalar projection of $\vec{a}$ on $\vec{b}$ is $\frac{-14}{3 \times 13} \times 3 = -\frac{14}{13}$. The vector projection of $\vec{a}$ on $\vec{b}$ is equal to the scalar projection multiplied by the unit vector in the direction of $\vec{b}$. So the vector projection is $-\frac{14}{13} \times \frac{1}{13}(3, -4, 12) = \left(-\frac{52}{169}, \frac{56}{169}, -\frac{168}{169}\right)$.

c. The scalar projection of $\vec{b}$ on $\vec{a}$ is equal to $|\vec{b}|\cos(\theta)$, where θ is the angle between the two vectors. So from the above work, $\cos(\theta) = \frac{-14}{3 \times 13}$ and $|\vec{b}| = 13$, so the scalar projection of $\vec{a}$ on $\vec{b}$ is $\frac{-14}{3 \times 13} \times 13 = -\frac{14}{3}$. The vector projection of $\vec{b}$ on $\vec{a}$ is equal to the scalar projection multiplied by the unit vector in the direction of $\vec{a}$. So the vector projection is $-\frac{14}{3} \times \frac{1}{3}(2, -1, -2) = \left(-\frac{28}{9}, \frac{14}{9}, \frac{28}{9}\right)$.

2. a. Since the normal of the first plane is (4, 2, 6) and the normal of the second is $(1, -1, 1)$, which are not scalar multiples of each other, there is a line of intersection between the planes.

The next step is to use the first and second equations to find an equation with a zero for the coefficient of x. The first equation minus four times the second equation yields $0x + 6y + 2z + 6 = 0$. We may divide by two to simplify, so $3y + z + 3 = 0$. If we let $y = t$, then $3t + z + 3 = 0$, or $z = -3 - 3t$. Substituting these into the second equation yields $x - (t) + (-3 - 3t) - 5 = 0$ or $x = 8 + 4t$.

So the equation of the line in parametric form is $x = 8 + 4t$, $y = t$, $z = -3 - 3t$, $t \in \mathbf{R}$.

To check that this is correct, we substitute in the solution to both initial equations

$$4x + 2y + 6z - 14 = 4(8 + 4t) + 2(t) + 6(-3 - 3t) - 14 = 0$$

and $x - y + z - 5$

$$= (8 + 4t) - (t) + (-3 - 3t) - 5 = 0.$$

Hence the line given by the parametric equation above is the line of intersection for the planes.

b. The angle between two planes is the same as the angle between their corresponding normal vectors.

$$|(4, 2, 6)| = \sqrt{4^2 + 2^2 + 6^2} = \sqrt{56}$$

$$|(1, -1, 1)| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$(4, 2, 6) \cdot (1, -1, 1) = 8$, so the angle between the planes is $\cos^{-1}\left(\frac{8}{\sqrt{3}\sqrt{56}}\right) \doteq 51.9°$.

3. a. We have that $\cos(60°) = \frac{\vec{x} \cdot \vec{y}}{|\vec{x}||\vec{y}|}$. Also since $\vec{x}$ and $\vec{y}$ are unit vectors, $|\vec{x}| = |\vec{y}| = 1$, and moreover $\cos(60°) = \frac{1}{2}$. So $\vec{x} \cdot \vec{y} = \frac{\vec{x} \cdot \vec{y}}{1 \times 1} = \frac{1}{2}$.

b. Scalar multiples can be brought out to the front of dot products. Hence $2\vec{x} \cdot 3\vec{y} = (2)(3)(\vec{x} \cdot \vec{y})$, and so by part **a.**, $2\vec{x} \cdot 3\vec{y} = 2 \times 3 \times \frac{1}{2} = 3$.

c. The dot product is distributive,

so $(2\vec{x} - \vec{y}) \cdot (\vec{x} + 3\vec{y})$

$$= 2\vec{x} \cdot (\vec{x} + 3\vec{y}) - \vec{y} \cdot (\vec{x} + 3\vec{y})$$

$$= 2\vec{x} \cdot \vec{x} + 2\vec{x} \cdot 3\vec{y} - \vec{y} \cdot \vec{x} - \vec{y} \cdot 3\vec{y}$$

$$= 2\vec{x} \cdot \vec{x} + 2\vec{x} \cdot 3\vec{y} - \vec{x} \cdot \vec{y} - 3\vec{y} \cdot \vec{y}$$

Since $\vec{x}$ and $\vec{y}$ are unit vectors, $\vec{x} \cdot \vec{x} = \vec{y} \cdot \vec{y} = 1$, and so by using the values found in part **a.** and **b.**,

$$(2\vec{x} - \vec{y}) \cdot (\vec{x} + 3\vec{y}) = 2(1) + (3) - \left(\tfrac{1}{2}\right) - 3(1) = \frac{3}{2}$$

4. a. $2(\vec{i} - 2\vec{j} + 3\vec{k}) - 4(2\vec{i} + 4\vec{j} + 5\vec{k}) - (\vec{i} - \vec{j})$

$$= 2\vec{i} - 4\vec{j} + 6\vec{k} - 8\vec{i} - 16\vec{j} - 20\vec{k} - \vec{i} + \vec{j}$$

$$= -7\vec{i} - 19\vec{j} - 14\vec{k}$$

b. $-2(3\vec{i} - 4\vec{j} - 5\vec{k}) \cdot (2\vec{i} + 3\vec{k}) + 2\vec{i} \cdot (3\vec{j} - 2\vec{k})$
$= -2(3\vec{i} - 4\vec{j} - 5\vec{k}) \cdot (2\vec{i} + 0\vec{j} + 3\vec{k})$
$+ 2(\vec{i} + 0\vec{j} + 0\vec{k}) \cdot (0\vec{i} + 3\vec{j} - 2\vec{k})$
$= -2(3(2) - 4(0) - 5(3)) + 2(1(0)$
$+ 0(3) + 0(-2))$
$= -2(-9) + 2(0)$
$= 18$

5. The direction vectors for the positive x-axis, y-axis, and z-axis are (1, 0, 0), (0, 1, 0), and (0, 0, 1), respectively.

$|(4, -2, -3)| = \sqrt{4^2 + (-2)^2 + (-3)^2}$
$= \sqrt{29},$

and $|(1, 0, 0)| = |(0, 1, 0)|$
$= |(0, 0, 1)|$
$= \sqrt{1}$
$= 1.$

$(4, -2, -3) \cdot (1, 0, 0) = 4$, so the angle the vector makes with the x-axis is $\cos^{-1}\left(\frac{4}{1\sqrt{29}}\right) \doteq 42.0°$.

$(4, -2, -3) \cdot (0, 1, 0) = -2$, so the angle the vector makes with the y-axis is $\cos^{-1}\left(\frac{-2}{1\sqrt{29}}\right) = 111.8°$.

$(4,-2,-3) \cdot (0, 0, 1) = -3$, hence the angle the vector makes with the z-axis is $\cos^{-1}\left(\frac{-3}{1\sqrt{29}}\right) \doteq 123.9°$.

6. a. $\vec{a} \times \vec{b} = (1, -2, 3) \times (-1, 1, 2)$
$= (-2(2) - 3(1), 3(-1) - 1(2),$
$1(1) - (-2)(-1))$
$= (-7, -5, -1)$

b. By the scalar law for vector multiplication,
$2\vec{a} \times 3\vec{b} = 2(3)(\vec{a} \times \vec{b})$
$= 6(\vec{a} \times \vec{b})$
$= 6(-7, -5, -1) = (-42, -30, -6)$

c. The area of a parallelogram determined by $\vec{a}$ and $\vec{b}$ is equal to the magnitude of the cross product of $\vec{a}$ and $\vec{b}$.
$A =$ area of parallelogram
$= |\vec{a} \times \vec{b}|$
$= |(-7, -5, -1)|$
$= \sqrt{(-7)^2 + (-5)^2 + (-1)^2}$
$\doteq 8.66$ square units

d. $(\vec{b} \times \vec{a}) = -(\vec{a} \times \vec{b})$
$= -(-7, -5, -1)$
$= (7, 5, 1)$

So $\vec{c} \cdot (\vec{b} \times \vec{a}) = (3, -4, -1) \cdot (7, 5, 1)$
$= 3(7) - 4(5) - 1(1)$
$= 0$

7. A unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ can be determined from any vector perpendicular to both $\vec{a}$ and $\vec{b}$. $\vec{a} \times \vec{b}$ is a vector perpendicular to both $\vec{a}$ and $\vec{b}$.

$\vec{a} \times \vec{b} = (1, -1, 1) \times (2, -2, 3)$
$= (-1(3) - 1(-2), 1(2) - 1(3),$
$1(-2) - (-1)(2))$
$= (-1, -1, 0)$
$|\vec{a} \times \vec{b}| = |(-1, -1, 0)|$
$= \sqrt{(-1)^2 + (-1)^2 + 0^2}$
$= \sqrt{2}$

So $\frac{1}{\sqrt{2}}(-1, -1, 0) = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$ is an unit vector perpendicular to both $\vec{a}$ and $\vec{b}$. $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ is another.

8. a. Answers may vary. For example:
A direction vector for the line is $\overrightarrow{AB}$.
$\overrightarrow{AB} = (1, 2, 3) - (2, -3, 1)$
$= (-1, 5, 2)$
Since $A(2, -3, 1)$ is a point on the line, $\vec{r} = (2, -3, 1) + t(-1, 5, 2), t \in \mathbf{R}$, is a vector equation for a line and the corresponding parametric equation is $x = 2 - t, y = -3 + 5t, z = 1 + 2t$, $t \in \mathbf{R}$.

b. If the x-coordinate of a point on the line is 4, then $2 - t = 4$, or $t = -2$. At $t = -2$, the point on the line is $(2, -3, 1) - 2(-1, 5, 2) = (4, -13, -3)$. Hence $C(4, -13, -3)$ is a point on the line.

9. The direction vector of the first line is $(-1, 5, 2)$, while the direction vector for the second line is $(1, -5, -2) = -(-1, 5, 2)$. So the direction vectors for the line are collinear. Hence the lines are parallel. The lines coincide if and only if for any point on the first line and any point on the second line, the vector connecting the two points is a multiple of the direction vector for the lines.
(2, 0, 9) is a point on the first line and (3, −5, 10) is a point on the second line.
$(2, 0, 9) - (3, -5, 10) = (-1, 5, -1) \neq k(-1, 5, 2)$ for any $k \in \mathbf{R}$. Hence the lines are parallel and distinct.

10. The direction vector for the parallel line is (0, 1, 1). Since parallel lines have collinear direction vectors, (0, 1, 1) can be used as a direction vector for the line. Since (0, 0, 4) is a point on the line, $\vec{r} = (0, 0, 4) + t(0, 1, 1), t \in \mathbf{R}$, is a vector equation for a line and the corresponding parametric equation is $x = 0, y = t, z = 4 + t, t \in \mathbf{R}$.

11. The line is parallel to the plane if and only if the direction vector for the line is perpendicular to the normal vector for the plane. The normal vector for the plane is $(2, 3, c)$. The direction vector for the line is (2, 3, 1). The vectors are perpendicular if and only if the dot product between the two is zero.

$(2, 3, c) \cdot (2, 3, 1) = 2(2) + 3(3) + c(1)$
$= 13 + c$

So if $c = -13$, then the dot product of normal vector and the direction vector is zero. Hence for $c = -13$, the line and plane are parallel.

12. First put the line in its corresponding parametric form. (3, 1, 5) is a direction vector and $(2, -5, 3)$ is the origin point, so a parametric equation for the line is $x = 2 + 3s$, $y = -5 + s$, $z = 3 + 5s$, $s \in \mathbf{R}$. If we substitute these coordinates into the equation of the plane, we may find the s value where the line intersects the plane.

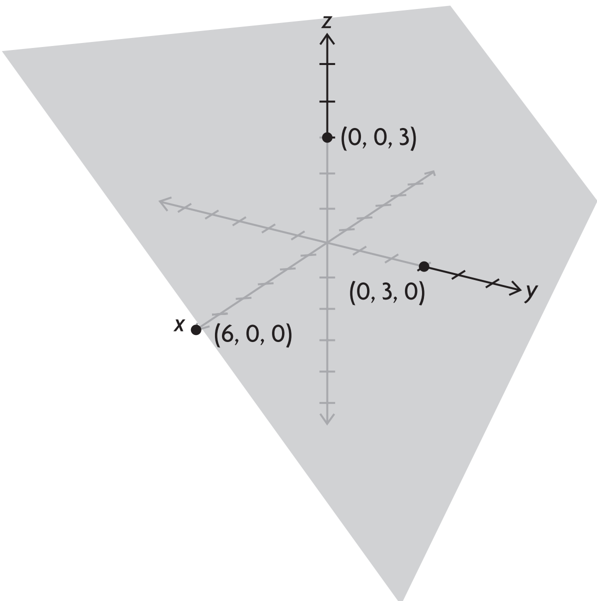
$5x + y - 2z + 2$
$= 5(2 + 3s) + (-5 + s) - 2(3 + 5s) + 2$
$= 10 + 15s + \ - 5 + s - 6 - 10s + 2$
$= 1 + 6s$

So if $5x + y - 2z + 2 = 0$, then $1 + 6s = 0$ or $s = -\frac{1}{6}$. At $s = -\frac{1}{6}$, the point on the line is $\left(\frac{3}{2}, -\frac{31}{6}, \frac{13}{6}\right)$. To check that this point is also on the plane, we substitute the x, y, z values into the plane equation and check that it equals zero.

$$5x + y - 2z + 2 = 5\left(\frac{3}{2}\right) + \left(-\frac{31}{6}\right) - 2\left(\frac{13}{6}\right) + 2$$
$$= 0$$

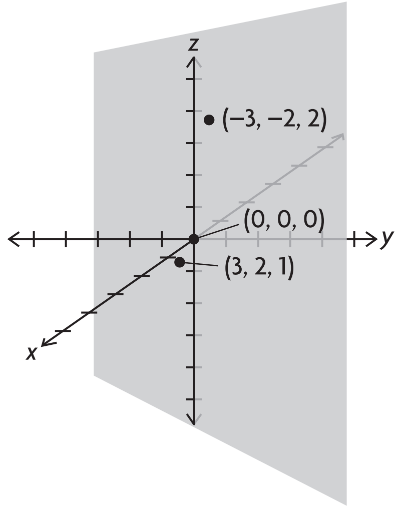
Hence $\left(\frac{3}{2}, -\frac{31}{6}, \frac{13}{6}\right)$ is the point of intersection between the line and the plane.

13. a.



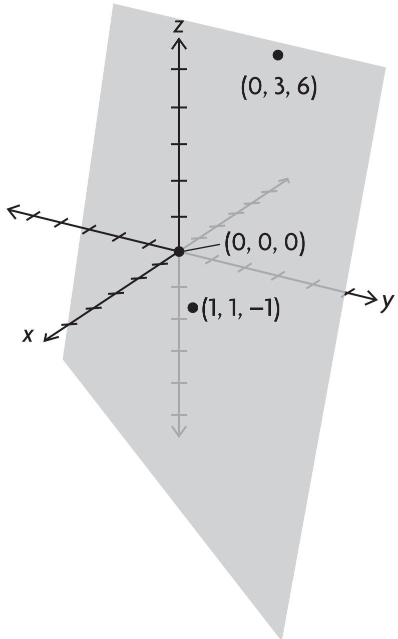
Two direction vectors are:
$(0, 3, 0) - (0, 0, 3) = (0, 3, -3)$
and
$(6, 0, 0) - (0, 0, 3) = (6, 0, -3)$.

b.



Two direction vectors are:
$(-3, -2, 2) - (0, 0, 0) = (-3, -2, 2)$
and
$(3, 2, 1) - (0, 0, 0) = (3, 2, 1)$.

c.



Two direction vectors are:
$(0, 3, 6) - (0, 0, 0) = (0, 3, 6)$
and
$(1, 1, -1) - (0, 0, 0) = (1, 1, -1)$.

14. The plane is the right bisector joining $P(1, -2, 4)$ and its image. The line connecting the two points has a direction vector equal to that of the normal vector for the plane. The normal vector for the plane is $(2, -3, -4)$. So the line connecting the two points is $(1, -2, 4) + t(2, -3, -4)$, $t \in \mathbf{R}$, or in

corresponding parametric form is $x = 1 + 2t$, $y = -2 - 3t$, $z = 4 - 4t$, $t \in \mathbf{R}$.
The intersection of this line and the plane is the bisector between P and its image. To find this point we substitute the parametric equation into the plane equation and solve for t.

$$\begin{aligned} &2x - 3y - 4z + 66 \\ &= 2(1 + 2t) - 3(-2 - 3t) - 4(4 - 4t) + 66 \\ &= 2 + 4t + 6 + 9t - 16 + 16t + 66 \\ &= 58 + 29t \end{aligned}$$

So if $2x - 3y - 4z + 66 = 0$, then $58 + 29t = 0$, or $t = -2$.
So the point of intersection is occurs at $t = -2$, since the origin point is P and the intersection occurs at the midpoint of the line connecting P and its image, the image point occurs at $t = 2 \times (-2) = -4$.
So the image point is at $x = 1 + 2(-4) = -7$, $y = -2 - 3(-4) = 10$, $z = 4 - 4(-4) = 20$.
So the image point is $(-7, 10, 20)$.

15. Let (a, b, c) be the direction vector for this line. So a line equation is $\vec{r} = (1, 0, 2) + t(a, b, c)$, $t \in \mathbf{R}$.
Since (1, 0, 2) is not on the other line, we may choose a, b, and c such that the intersection occurs at $t = 1$. Since the line is supposed to intersect the given line at a right angle, the direction vectors should be perpendicular. The direction vectors are perpendicular if and only if their dot product is zero. The direction vector for the given line is (1, 1, 2).
$(a, b, c) \cdot (1, 1, 2) = a + b + 2c = 0$, so $b = -a - 2c$.
Also $(1, 0, 2) + (a, b, c) = (1 + a, b, 2 + c)$ is the point of intersection.
By substituting for b,
$(1 + a, b, 2 + c) = (1 + a, -a - 2c, 2 + c)$.
So for some s value,

$$\begin{aligned} x &= -2 + s = 1 + a \\ y &= 3 + s = -a - 2c \\ z &= 4 + 2s = 2 + c \end{aligned}$$

Subtracting the first equation from the second yields the equation, $5 + 0s = -2a - 2c - 1$.
Simplifying this gives $6 = -2a - 2c$ or just $a + c = -3$.
Subtracting twice the first equation from the third yields the equation, $8 = -2a + c$.
So $a + c = -3$ and $-2a + c = 8$, which is two equations with two unknowns. Twice the first plus the second equations gives $0a + 3c = 2$ or $c = \frac{2}{3}$.
Solving back for a gives $-\frac{11}{3}$ and since $b = -a - 2c$, $b = \frac{7}{3}$. Since $a + b + 2c = 0$, the direction vectors, (1, 1, 2) and (a, b, c) are perpendicular. A direction vector for the line is $(-11, 7, 2)$.
We need to check that $(1, 0, 2) + (a, b, c) = \left(\frac{-8}{3}, \frac{7}{3}, \frac{8}{3}\right)$ is a point on the given line.
$x = -2 + s = -\frac{8}{3}$, at $s = -\frac{2}{3}$. The point on the given line at $s = -\frac{2}{3}$ is $\left(\frac{-8}{3}, \frac{7}{3}, \frac{8}{3}\right)$. Hence $\vec{q} = (1, 0, 2) + t(-11, 7, 2)$, $t \in \mathbf{R}$, is a line that intersects the given line at a right angle.

16. a. The Cartesian equation is found by taking the cross product of the two direction vectors, $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$$\begin{aligned} \overrightarrow{AB} &= (-2, 0, 0) - (1, 2, 3) \\ &= (-3, -2, -3) \\ \overrightarrow{AC} &= (1, 4, 0) - (1, 2, 3) = (0, 2, -3) \\ \overrightarrow{AB} \times \overrightarrow{AC} &= (-2(-3) - (-3)(2), \\ &\quad -3(0) - (-3)(-3), \\ &\quad -3(2) - (-2)(0)) \\ &= (12, -9, -6) \end{aligned}$$

So $= (12, -9, -6)$ is a normal vector for the plane, so the plane has the form $12x - 9y - 6z + D = 0$, for some constant D. To find D, we know that $A(1, 2, 3)$ is a point on the plane, so $12(1) - 9(2) - 6(3) + D = 0$. So $-24 + D = 0$, or $D = 24$. So the Cartesian equation for the plane is $12x - 9y - 6z + 24 = 0$.
b. Substitute into the formula to determine distance between a point and a plane. So the distance, d, of (0, 0, 0) to the plane $12x - 9y - 6z + 24 = 0$ is equal to $\frac{|12(0) - 9(0) - 6(0) + 24|}{\sqrt{12^2 + (-9)^2 + (-6)^2}}$.
So $d = \frac{24}{\sqrt{261}} \doteq 1.49$.

17. a. $(3, -5, 4)$ is a normal vector for the plane, so the plane has the form $3x - 5y + 4z + D = 0$, for some constant D. To find D, we know that $A(-1, 2, 5)$ is a point on the plane, so $3(-1) - 5(2) + 4(5) + D = 0$. So $7 + D = 0$, or $D = -7$. So the Cartesian equation for the plane is $3x - 5y + 4z - 7 = 0$.
b. Since the plane is perpendicular to the line connecting (2, 1, 8) and $(1, 2, -4)$, a direction vector for the line acts as a normal vector for the plane. So $(2, 1, 8) - (1, 2, -4) = (1, -1, 12)$ is a normal vector for the plane. So the plane has the form $x - y + 12z + D = 0$, for some constant D. To find D, we know that $K(4, 1, 2)$ is a point on the plane, so $(4) - (1) + 12(2) + D = 0$. So $27 + D = 0$, or $D = -27$. So the Cartesian equation for the plane is $x - y + 12z - 27 = 0$.

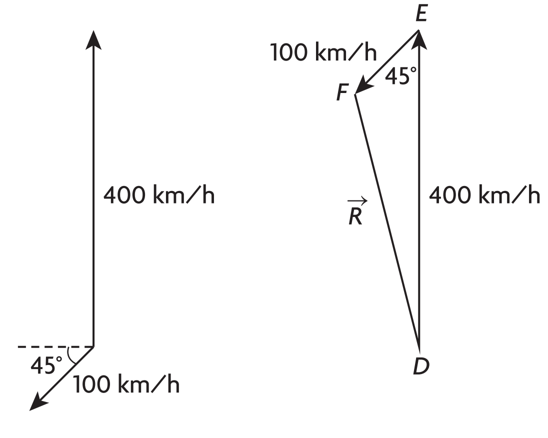
c. Since the plane is perpendicular to the z-axis, a direction vector for the z-axis acts as a normal vector for the plane. Hence (0, 0, 1) is a normal vector for the plane. So the plane has the form $z + D = 0$, for some constant D. To find D, we know that $(3, -1, 3)$ is a point on the plane, so
$0(3) + 0(-1) + (3) + D = 0$. So $3 + D = 0$, or $D = -3$. So the Cartesian equation for the plane is $z - 3 = 0$.

d. The Cartesian equation can be found by taking the cross product of the two direction vectors for the plane. Since $(3, 1, -2)$ and $(1, 3, -1)$ are two points on the plane
$(3, 1, -2) - (1, 3, -1) = (2, -2, -1)$ is a direction vector for the plane. Since the plane is parallel to the y-axis, (0, 1, 0) is also a direction vector for the plane.

$$(2, -2, -1) \times (0, 1, 0) = (-2(0) - (-1)(1), (-1)(0) - (2)(0), 2(1) - (-2)(0))$$
$$= (1, 0, 2)$$

So $(1, 0, 2)$ is a normal vector for the plane, so the plane has the form $x + 0y + 2z + D = 0$, for some constant D. To find D, we know that $(3, 1, -2)$ is a point on the plane, so
$(3) + 0(1) + 2(-2) + D = 0$. So $-1 + D = 0$, or $D = 1$. So the Cartesian equation for the plane is $x + 2z + 1 = 0$.

18.

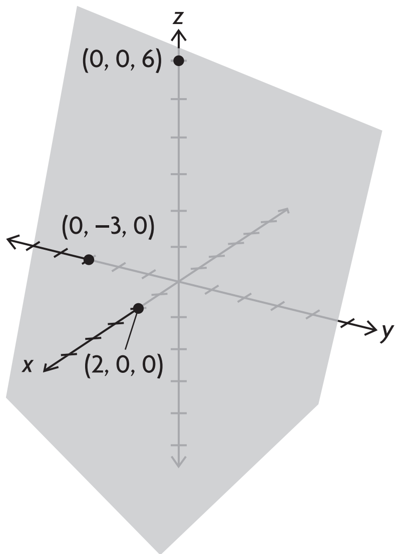


Position Diagram   Vector Diagram

From the triangle DEF and the cosine law, we have
$$|\vec{R}|^2 = 400^2 + 100^2 - 2(400)(100)\cos(45°)$$
$$\doteq 336.80 \text{ km/h}.$$
To find the direction of the vector, the sine law is applied.
$$\frac{\sin \angle DEF}{|\vec{R}|} = \frac{\sin \angle EDF}{100}$$
$$\frac{\sin 45°}{336.80} \doteq \frac{\sin \angle EDF}{100}.$$
$$\sin \angle EDF \doteq \frac{\sin 45°}{336.80} \times 100.$$
$$\sin \angle EDF \doteq 0.2100.$$
Thus $\angle EDF \doteq 12.1°$, so the resultant velocity is 336.80 km/h, N 12.1° W.

19. a. The simplest way is to find the parametric equation, then find the corresponding vector equation. If we substitute $x = s$ and $y = t$ and solve for z, we obtain $3s - 2t + z - 6 = 0$ or $z = 6 - 3s + 2t$. This yields the parametric equations $x = s$, $y = t$, and $z = 6 - 3s + 2t$. So the corresponding vector equation is $\vec{r} = (0, 0, 6) + s(1, 0, -3) + t(0, 1, 2)$, $s, t \in \mathbf{R}$. To check that this is correct, find the Cartesian equation corresponding to the above vector equation and see if it is equivalent to the Cartesian equation given in the problem. A normal vector to this plane is the cross product of the two directional vectors.
$$\vec{n} = (1, 0, -3) \times (0, 1, 2) = (0(2) - (-3)(1), -3(0) - 1(2), 1(1) - 0(0))$$
$$= (3, -2, 1)$$
So $(3, -2, 1)$ is a normal vector for the plane, so the plane has the form $3x - 2y + z + D = 0$, for some constant D. To find D, we know that (0, 0, 6) is a point on the plane, so $3(0) - 2(0) + (6) + D = 0$.
So $6 + D = 0$, or $D = -6$. So the Cartesian equation for the plane is $3x - 2y + z - 6 = 0$. Since this is the same as the initial Cartesian equation, the vector equation for the plane is correct.

b.



20. a. The angle, θ, between the plane and the line is the complementary angle of the angle between the direction vector of the line and the normal vector for the plane. The direction vector of the line is $(2, -1, 2)$ and the normal vector for the plane is $(1, 2, 1)$.

$$|(2, -1, 2)| = \sqrt{2^2 + (-1)^2 + 2^2}$$
$$= \sqrt{9}$$
$$= 3.$$
$$|(1, 2, 1)| = \sqrt{1^2 + 2^2 + 1^2}$$
$$= \sqrt{6}$$
$$(2, -1, 2) \cdot (1, 2, 1) = 2(1) - 1(2) + 2(1) = 2$$

So the angle between the normal vector and the direction vector is $\cos^{-1}\left(\frac{2}{3\sqrt{6}}\right) \doteq 74.21°$. So

$\theta \doteq 90° - 74.21° = 15.79°$.

To the nearest degree, $\theta = 16°$.

b. The two planes are perpendicular if and only if their normal vectors are also perpendicular.

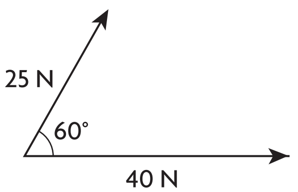
A normal vector for the first plane is $(2, -3, 1)$ and a normal vector for the second plane is $(4, -3, -17)$. The two vectors are perpendicular if and only if their dot product is zero.

$$(2, -3, 1) \cdot (4, -3, -17) = 2(4) - 3(-3) + 1(-17) = 0.$$

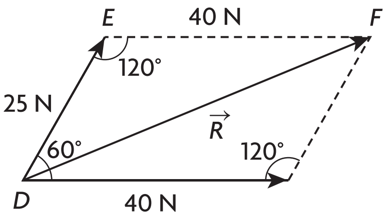
Hence the normal vectors are perpendicular. Thus the planes are perpendicular.

c. The two planes are parallel if and only if their normal vectors are also parallel. A normal vector for the first plane is $(2, -3, 2)$ and a normal vector for the second plane is $(2, -3, 2)$. Since both normal vectors are the same, the planes are parallel. Since $2(0) - 3(-1) + 2(0) - 3 = 0$, the point $(0, -1, 0)$ is on the second plane. Yet since $2(0) - 3(-1) + 2(0) - 1 = 2 \neq 0$, $(0, -1, 0)$ is not on the first plane. Thus the two planes are parallel but not coincident.

21.



Position diagram



Vector diagram

From the triangle DEF and the cosine law, we have

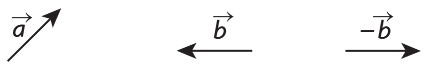
$$|\vec{R}|^2 = 40^2 + 25^2 - 2(40)(25)\cos(120°)$$
$$\doteq 56.79 \text{ N}.$$

To find the direction of the vector, the sine law is applied.

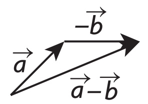
$$\frac{\sin \angle DEF}{|\vec{R}|} = \frac{\sin \angle EDF}{100}$$
$$\frac{\sin 120°}{56.79} \doteq \frac{\sin \angle EDF}{40}.$$
$$\sin \angle EDF \doteq \frac{\sin 120°}{56.79} \times 40.$$
$$\sin \angle EDF \doteq 0.610.$$

Thus $\angle EDF \doteq 37.6°$, so the resultant force is approximately 56.79 N, 37.6° from the 25 N force towards the 40 N force. The equilibrant force has the same magnitude as the resultant, but it is in the opposite direction. So the equilibrant is approximately 56.79 N, $180° - 37.6° = 142.4°$ from the 25 N force away from the 40 N force.

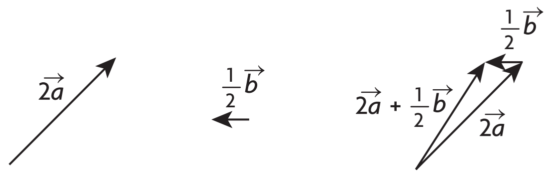
22.



a.



b.



23. a. The unit vector in the same direction of $\vec{a}$ is simply $\vec{a}$ divided by the magnitude of $\vec{a}$.

$$|\vec{a}| = \sqrt{6^2 + 2^2 + (-3)^2}$$
$$= \sqrt{49}$$
$$= 7$$

So the unit vector in the same direction of $\vec{a}$ is

$\frac{1}{|\vec{a}|}\vec{a} = \frac{1}{7}(6, 2, -3) = \left(\frac{6}{7}, \frac{2}{7}, -\frac{3}{7}\right)$.

b. The unit vector in the opposite direction of $\vec{a}$ is simply the negative of the unit vector found in part a. So the vector is $-\left(\frac{6}{7}, \frac{2}{7}, -\frac{3}{7}\right) = \left(-\frac{6}{7}, -\frac{2}{7}, \frac{3}{7}\right)$.

24. a. Since $OBCD$ is a parallelogram, the point C occurs at $(-1, 7) + (9, 2) = (8, 9)$. So $\overrightarrow{OC}$ is one vector equivalent to a diagonal and $\overrightarrow{BD}$ is the other.

$$\overrightarrow{OC} = (8, 9) - (0, 0) = (8, 9)$$
$$\overrightarrow{BD} = (9, 2) - (1, 7) = (10, -5)$$

b. $|(8, 9)| = \sqrt{8^2 + 9^2}$
$= \sqrt{145}$
$|(10, -5)| = \sqrt{10^2 + (-5)^2}$
$= \sqrt{125}$
$(8, 9) \cdot (10, -5) = 8(10) + 9(-5)$
$= -35$
So the angle between these diagonals is
$\cos^{-1}\left(\frac{-35}{\sqrt{145}\sqrt{125}}\right) \doteq 74.9°$.
c. $\overrightarrow{OB} = (-1, 7)$ and $\overrightarrow{OD} = (9, 2)$
$|(-1, 7)| = \sqrt{(-1)^2 + 7^2}$.
$= \sqrt{50}$
$|(9, 2)| = \sqrt{9^2 + 2^2}$
$= \sqrt{85}$
$(-1, 7) \cdot (9, 2) = -(9) + 7(2)$
$= 5$
So the angle between these diagonals is
$\cos^{-1}\left(\frac{5}{\sqrt{50}\sqrt{85}}\right) \doteq 85.6°$.
25. a. First step is to use the first equation to remove x from the second and third.
① $x - y + z = 2$
② $-x + y + 2z = 1$
③ $x - y + 4z = 5$
So we have
④ $0x + 0y + 3z = 3$, ① + ②
⑤ $0x + 0y + 3z = 3$, $-1 \times$ ① + ③
Hence $3z = 3$, or $z = 1$. Since both equations are the same, this implies that there are infinitely many solutions. Let $x = t$, then by substituting into the equation 2, we obtain
$-t + y + 2(1) = 1$, or $y = -1 + t$.
Hence the solution to these equations is $x = t$, $y = -1 + t$, $z = 1$, $t \in \mathbf{R}$.
b. First step is to use the first equation to remove x from the second and third.
① $-2x - 3y + z = -11$
② $x + 2y + z = 2$
③ $-x - y + 3z = -12$
So we have
④ $0x + 1y + 3z = -7$, ① + 2 × ②
⑤ $0x - 1y - 5z = 13$, ① − 2 × ③
Now the fourth and fifth equations are used to create a sixth equation where the coefficient of y is zero.
⑥ $0x + 0y - 2z = 6$, ④ + ⑤
So $-2z = 6$ or $z = -3$.

Substituting this into equation ④ yields,
$y + 3(-3) = -7$ or $y = 2$. Finally substitute z and y values into equation ② to obtain the x value.
$x + 2(2) + (-3) = 2$ or $x = 1$.
Hence the solution to these three equations is $(1, 2, -3)$.
c. First step is to notice that the second equation is simply twice the first equation.
① $2x - y + z = -1$
② $4x - 2y + 2z = -2$
③ $2x + y - z = 5$
So the solution to these equations is the same as the solution to just the first and third equations. Moreover since this is two equations with three unknowns, there will be infinitely many solutions.
④ $4x + 0y + 0z = 4$, ① + ③
Hence $4x = 4$ or $x = 1$. Let $y = t$ and solve for z using the first equation.
$2(1) - t + z = -1$, so $z = -3 + t$
Hence the solution to these equations is $x = 1$, $y = t$, $z = -3 + t$, $t \in \mathbf{R}$.
d. First step is to notice that the second equations is simply twice the first and the third equation is simply -4 times the first equation.
① $x - y - 3z = 1$
② $2x - 2y - 6z = 2$
③ $-4x + 4y + 12z = -4$
So the solution to these equations is the same as the solution to just the first equation. So the solution to these equations is a plane. To solve this in parametric equation form, simply let $y = t$ and $z = s$ and find the x value.
$x - t - 3s = 1$, or $x = 1 + t + 3s$
So the solution to these equations is $x = 1 + 3s + t$, $y = t$, $z = s$, $s, t \in \mathbf{R}$.
26. a. Since the normal of the first equation is $(1, -1, 1)$ and the normal of the second is $(1, 2, -2)$, which are not scalar multiples of each other, there is a line of intersection between the planes. The next step is to use the first and second equations to find an equation with a zero for the coefficient of x. The second equation minus the first equation yields $0x + 3y - 3z + 3 = 0$. We may divide by three to simplify, so $y - z + 1 = 0$. If we let $z = t$, then $y - t + 1 = 0$, or $y = -1 + t$. Substituting these into the first equation yields $x - (-1 + t) + t - 1 = 0$ or $x = 0$. So the equation of the line in parametric form is $x = 0$, $y = -1 + t$, $z = t$, $t \in \mathbf{R}$.

To check that this is correct, we substitute in the solution to both initial equations

$$x - y + z - 1 = (0) - (-1 + t) + (t) - 1$$
$$= 0$$

and

$$x + 2y - 2z + 2 = (0) + 2(-1 + t) - 2(t) + 2$$
$$= 0.$$

Hence the line given by the parametric equation above is the line of intersection for the planes.

b. The normal vector for the first plane is $(1, -4, 7)$, while the normal vector for the second plane is $(2, -8, 14) = 2(1, -4, 7)$. Hence the planes have collinear normal vectors, and so are parallel.

The second equation is equivalent to $x - 4y + 7z = 30$, since we may divide the equation by two. Since the constant on the right in the first equation is 28, while the constant on the right in the second equivalent equation is 30, these planes are parallel and not coincident. So there is no intersection.

c. The normal vector for the first equation is $(1, -1, 1)$, while the normal vector for the second equation is (2, 1, 1). Since the normal vectors are not scalar multiples of each other, there is a line of intersection between the planes.

The next step is to use the first and second equations to find an equation with a zero for the coefficient of x. The second equation minus twice the first equation yields $0x + 3y - z + 0 = 0$.

Solving for z yields, $z = 3y$. If we let $y = t$, then $z = 3(t) = 3t$.

Substituting these into the first equation yields $x - (t) + (3t) - 2 = 0$ or $x = 2 - 2t$. So the equation of the line in parametric form is $x = 2 - 2t$, $y = t$, $z = 3t$, $t \in \mathbf{R}$.

To check that this is correct, we substitute in the solution to both initial equations

$$x - y + z - 2 = (2 - 2t) - (t) + (3t) - 2$$
$$= 0$$

and

$$2x + y + z - 4 = 2(2 - 2t) + (t) + (3t) - 4$$
$$= 0.$$

Hence the line given by the parametric equation above is the line of intersection for the planes.

27. The angle, θ, between the plane and the line is the complementary angle of the angle between the direction vector of the line and the normal vector for the plane. The direction vector of the line is $(1, -1, 0)$ and the normal vector for the plane is $(2, 0, -2)$.

$$|(1, -1, 0)| = \sqrt{1^2 + (-1)^2 + 0^2}$$
$$= \sqrt{2}$$
$$|(2, 0, -2)| = \sqrt{2^2 + 0^2 + (-2)^2} = \sqrt{8}$$

$$(1, -1, 0) \cdot (2, 0, -2) = 1(2) - 1(0) + 0(-2) = 2$$

So the angle between the normal vector and the direction vector is $\cos^{-1}\left(\frac{2}{\sqrt{2}\sqrt{8}}\right) = 60°$. So $\theta = 90 - 60° = 30°$.

28. a. We have that $\cos(60°) = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$. Also since $\vec{a}$ and $\vec{b}$ are unit vectors, $|\vec{a}| = |\vec{b}| = 1$ and $\vec{a} \cdot \vec{a} = \vec{b} \cdot \vec{b} = 1$, and moreover $\cos(60°) = \frac{1}{2}$. So $\vec{a} \cdot \vec{b} = \frac{\vec{a} \cdot \vec{b}}{1 \times 1} = \frac{1}{2}$.

The dot product is distributive, so

$$(6\vec{a} + \vec{b}) \cdot (\vec{a} - 2\vec{b}) = 6\vec{a} \cdot (\vec{a} - 2\vec{b}) + \vec{b} \cdot (\vec{a} - 2\vec{b})$$
$$= 6\vec{a} \cdot \vec{a} + 6\vec{a} \cdot (-2\vec{b}) + \vec{b} \cdot \vec{a} + \vec{b} \cdot (-2\vec{b})$$
$$= 6\vec{a} \cdot \vec{a} - 12\vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{b} - 2\vec{b} \cdot \vec{b}$$
$$= 6(1) - 12\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) - 2(1)$$
$$= -\frac{3}{2}$$

b. We have that $\cos(60°) = \frac{\vec{x} \cdot \vec{y}}{|\vec{x}||\vec{y}|}$. Also since $|\vec{x}| = 3$, $|\vec{y}| = 4$, and $\cos(60°) = \frac{1}{2}$, $\vec{x} \cdot \vec{y} = \frac{1}{2}(4)(3) = 6$. Also $\vec{x} \cdot \vec{x} = |\vec{x}|^2 = 9$ and $\vec{y} \cdot \vec{y} = |\vec{y}|^2 = 16$.

The dot product is distributive, so

$$(4\vec{x} - \vec{y}) \cdot (2\vec{x} + 3\vec{y}) = 4\vec{x} \cdot (2\vec{x} + 3\vec{y}) - \vec{y} \cdot (2\vec{x} + 3\vec{y})$$
$$= 8\vec{x} \cdot \vec{x} + 12\vec{x} \cdot \vec{y} - 2\vec{y} \cdot \vec{x} - 3\vec{y} \cdot \vec{y}$$
$$= 8(9) + 12(6) - 2(6) - 3(16)$$
$$= 84$$

29. The origin, (0, 0, 0), and $(-1, 3, 1)$ are two points on this line. So $(-1, 3, 1)$ is a direction vector for this line and since the origin is on the line, a possible vector equation is $\vec{r} = t(-1, 3, 1)$, $t \in \mathbf{R}$. $(-1, 3, 1)$ is a normal vector for the plane. So the equation of the plane is $-x + 3y + z + D = 0$.

$(-1, 3, 1)$ is a point on the plane. Substitute the coordinates to determine the value of D.

$$1 + 9 + 1 + D = 0$$
$$D = -11$$

The equation of the plane is $-x + 3y + z - 11 = 0$.

30. The plane is the right bisector joining $P(-1, 0, 1)$ and its image. The line connecting the two points has a direction vector equal to that of the normal vector for the plane. The normal vector for the plane is $(0, 1, -1)$. So the line connecting the two points is $(-1, 0, 1) + t(0, 1, -1), t \in \mathbf{R}$, or in corresponding parametric form is $x = -1, y = t, z = 1 - t, t \in \mathbf{R}$. The intersection of this line and the plane is the bisector between P and its image. To find this point we plug the parametric equation into the plane equation and solve for t.

$$0x + y - z = 0(-1) + (t) - (1 - t)$$
$$= -1 + 2t$$

So if $y - z = 0$, then $-1 + 2t = 0$, or $t = \frac{1}{2}$.

So the point of intersection is occurs at $t = \frac{1}{2}$, since the origin point is P and the intersection occurs at the midpoint of the line connecting P and its image, the image point occurs at $t = 2 \times \frac{1}{2} = 1$. So the image point is at $x = -1, y = 1, z = 1 - (1) = 0$. So the image point is $(-1, 1, 0)$.

31. a. Thinking of the motorboat's velocity vector (without the influence of the current) as starting at the origin and pointing northward toward the opposite side of the river, the motorboat has velocity vector (0, 10) and the river current has velocity vector (4, 0). So the resultant velocity vector of the motorboat is

$$(0, 10) + (4, 0) = (4, 10)$$

To reach the other side of the river, the motorboat needs to cover a vertical distance of 2 km. So the hypotenuse of the right triangle formed by the marina, the motorboat's initial position, and the motorboat's arrival point on the opposite side of the river is represented by the vector

$$\frac{1}{5}(4, 10) = \left(\frac{4}{5}, 2\right)$$

(We multiplied by $\frac{1}{5}$ to create a vertical component of 2 in the motorboat's resultant velocity vector, the distance needed to cross the river.) Since this new vector has horizontal component equal to $\frac{4}{5}$, this means that the motorboat arrives $\frac{4}{5} = 0.8$ km downstream from the marina.

b. The motorboat is travelling at 10 km/h, and in part a. we found that it will travel along the vector $\left(\frac{4}{5}, 2\right)$. The length of this vector is

$$\left|\left(\frac{4}{5}, 2\right)\right| = \sqrt{\left(\frac{4}{5}\right)^2 + 2^2}$$
$$= \sqrt{4.64}$$

So the motorboat travels a total of $\sqrt{4.64}$ km to cross the river which, at 10 km/h, takes

$$\sqrt{4.64} \div 10 \doteq 0.2 \text{ hours}$$
$$= 12 \text{ minutes.}$$

32. a. Answers may vary. For example:

A direction vector for this line is

$$\overrightarrow{AB} = (6, 3, 4) - (2, -1, 3)$$
$$= (4, 4, 1)$$

So, since the point $B(6, 3, 4)$ is on this line, the vector equation of this line is

$\vec{r} = (6, 3, 4) + t(4, 4, 1), t \in \mathbf{R}$.

The equivalent parametric form is

$x = 6 + 4t$

$y = 3 + 4t$

$z = 4 + t, t \in \mathbf{R}$.

b. The line found in part a. will lie in the plane $x - 2y + 4z - 16 = 0$ if and only if both points $A(2, -1, 3)$ and $B(6, 3, 4)$ lie in this plane.

We verify this by substituting these points into the equation of the plane, and checking for consistency.

For A:

$$2 - 2(-1) + 4(3) - 16 = 0$$

For B:

$$6 - 2(3) + 4(4) - 16 = 0$$

Since both points lie on the plane, so does the line found in part a.

33. The wind velocity vector is represented by (16, 0), and the water current velocity vector is represented by (0, 12). So the resultant of these two vectors is $(16, 0) + (0, 12) = (16, 12)$.

Thinking of this vector with tail at the origin and head at point (16, 12), this vector forms a right triangle with vertices at points (0, 0), (0, 12), and (16, 12). Notice that

$$|(16, 12)| = \sqrt{16^2 + 12^2}$$
$$= \sqrt{400}$$
$$= 20$$

This means that the sailboat is moving at a speed of 20 km/h once we account for wind and water velocities. Also the angle, θ, this resultant vector makes with the positive y-axis satisfies

$$\cos\theta = \frac{12}{20}$$
$$\theta = \cos^{-1}\left(\frac{12}{20}\right)$$
$$\doteq 53.1°$$

So the sailboat is travelling in the direction N 53.1° E, or equivalently E 36.9° N.

34. Think of the weight vector for the crane with tail at the origin at head at $(0, -400)$ (we use one unit for every kilogram of mass). We need to express this weight vector as the sum of two vectors: one that is parallel to the inclined plane and pointing down this incline (call this vector $\vec{x} = (a, b)$), and one that is perpendicular to the inclined plane and pointing toward the plane (call this vector $\vec{y} = (c, d)$). The angle between $\vec{x}$ and $(0, -400)$ is 60° and the angle between $\vec{y}$ and $(0, -400)$ is 30°. Of course, $\vec{x}$ and $\vec{y}$ are perpendicular. Using the formula for dot product, we get

$$\vec{y} \cdot (0, -400) = |\vec{y}||(0, -400)|\cos 30°$$
$$-400d = 400\left(\frac{\sqrt{3}}{2}\right)\sqrt{c^2 + d^2}$$
$$-2d = \sqrt{3} \cdot \sqrt{c^2 + d^2}$$
$$4d^2 = 3(c^2 + d^2)$$
$$d^2 = 3c^2$$

So, since c is positive and d is negative (thinking of the inclined plane as moving upward from left to right as we look at it means that $\vec{y}$ points down and to the right), this last equation means that $\frac{d}{c} = -\sqrt{3}$

So a vector in the same direction as $\vec{y}$ is $(1, -\sqrt{3})$. We can find the length of $\vec{y}$ by computing the scalar projection of $(0, -400)$ on $(1, -\sqrt{3})$, which equals

$$\frac{(0, -400) \cdot (1, -\sqrt{3})}{|(1, -\sqrt{3})|} = \frac{400\sqrt{3}}{2}$$
$$= 200\sqrt{3}$$

That is, $|\vec{y}| = 200\sqrt{3}$. Now we can find the length of $\vec{x}$ as well by using the fact that

$$|\vec{x}|^2 + |\vec{y}|^2 = |(0, -400)|^2$$
$$|\vec{x}|^2 + (200\sqrt{3})^2 = 400^2$$
$$|\vec{x}| = \sqrt{160\,000 - 120\,000}$$
$$= \sqrt{40\,000}$$
$$= 200$$

So we get that
$|\vec{x}| = 200$ and $|\vec{y}| = 200\sqrt{3}$. This means that the component of the weight of the mass parallel to the inclined plane is

$$9.8 \times |\vec{x}| = 9.8 \times 200$$
$$= 1960 \text{ N},$$

and the component of the weight of the mass perpendicular to the inclined plane is

$$9.8 \times |\vec{y}| = 9.8 \times 200\sqrt{3}$$
$$\doteq 3394.82 \text{ N}.$$

35. a. True; all non-parallel pairs of lines intersect in exactly one point in R^2. However, this is not the case for lines in R^3 (skew lines provide a counterexample).

b. True; all non-parallel pairs of planes intersect in a line in R^3.

c. True; the line $x = y = z$ has direction vector $(1, 1, 1)$, which is not perpendicular to the normal vector $(1, -2, 2)$ to the plane $x - 2y + 2z = k$, k any constant. Since these vectors are not perpendicular, the line is not parallel to the plane, and so they will intersect in exactly one point.

d. False; a direction vector for the line $\frac{x}{2} = y - 1 = \frac{z + 1}{2}$ is (2, 1, 2). A direction vector for the line $\frac{x - 1}{-4} = \frac{y - 1}{-2} = \frac{z + 1}{-2}$ is $(-4, -2, -2)$, or (2, 1, 1) (which is parallel to $(-4, -2, -2)$). Since (2, 1, 2) and (2, 1, 1) are obviously not parallel, these two lines are not parallel.

36. a. A direction vector for

$$L_1: x = 2, \frac{y - 2}{3} = z$$

is (0, 3, 1), and a direction vector for

$$L_2: x = y + k = \frac{z + 14}{k}$$

is $(1, 1, k)$. But (0, 3, 1) is not a nonzero scalar multiple of $(1, 1, k)$ for any k since the first component of (0, 3, 1) is 0. This means that the direction vectors for L_1 and L_2 are never parallel, which means that these lines are never parallel for any k.

b. If L_1 and L_2 intersect, in particular their x-coordinates will be equal at this intersection point. But $x = 2$ always in L_1 so we get the equation

$$2 = y + k$$
$$y = 2 - k$$

Also, from L_1 we know that $z = \frac{y - 2}{3}$, so substituting this in for z in L_2 we get

$$2k = z + 14$$

$$2k = \frac{y - 2}{3} + 14$$

$$3(2k - 14) = y - 2$$

$$y = 6k - 40$$

So since we already know that $y = 2 - k$, we now get

$$2 - k = 6k - 40$$

$$7k = 42$$

$$k = 6$$

So these two lines intersect when $k = 6$. We have already found that $x = 2$ at this intersection point, but now we know that

$$y = 6k - 40$$

$$= 6(6) - 40$$

$$= -4$$

$$z = \frac{y - 2}{3}$$

$$= \frac{-4 - 2}{3}$$

$$= -2$$

So the point of intersection of these two lines is $(2, -4, -2)$, and this occurs when $k = 6$.

Vector Appendix

Gaussian Elimination, pp. 588–590

1. a. First write the system in matrix form (omitting the variables and using only coefficients of each equation). The coefficients of the unknowns are entered in columns on the left side of a matrix with a vertical line separating the coefficients from the numbers on the right side.

① $x + 2y - z = -1$
② $-x + 3y - 2z = -1$
③ $3y - 2z = -3$

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & -1 \\ -1 & 3 & -2 & -1 \\ 0 & 3 & -2 & -3 \end{array}\right]$$

b. ① $2x - z = 1$
② $2y - z = 16$
③ $-3x + y = 10$

$$\left[\begin{array}{rrr|r} 2 & 0 & -1 & 1 \\ 0 & 2 & -1 & 16 \\ -3 & 1 & 0 & 10 \end{array}\right]$$

c. ① $2x - y - z = -2$
② $x - y + 4z = -1$
③ $-x - y = 13$

$$\left[\begin{array}{rrr|r} 2 & -1 & -1 & -2 \\ 1 & -1 & 4 & -1 \\ -1 & -1 & 0 & 13 \end{array}\right]$$

2. Answers may vary. For example:

$$\left[\begin{array}{rr|r} 2 & 3 & 0 \\ 3 & -1 & 1 \end{array}\right]$$

$$\left[\begin{array}{rr|r} 1 & 1.5 & 0 \\ 3 & -1 & 1 \end{array}\right] \frac{1}{2}\text{(row 1)}$$

$$\left[\begin{array}{rr|r} 1 & 1.5 & 0 \\ 0 & -5.5 & 1 \end{array}\right] \begin{array}{l} \\ -3\text{ (row 1)} + \text{row 2} \end{array}$$

$$\left[\begin{array}{rr|r} 2 & 3 & 0 \\ 0 & -5.5 & 1 \end{array}\right] \begin{array}{l} 2\text{ (row 1)} \\ \\ \end{array}$$

The last two matrices are both in row-echelon form and are equivalent.

3. Answers may vary. For example:

$$\left[\begin{array}{rrr|r} 2 & 1 & 6 & 0 \\ 0 & -2 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{rrr|r} 1 & 0.5 & 3 & 0 \\ 0 & -2 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{array}\right] \frac{1}{2}\text{(row 1)}$$

$$\left[\begin{array}{rrr|r} 1 & 0.5 & 3 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -0.5 & -9 & 1 \end{array}\right] \begin{array}{l} \\ \\ -3\text{ (row 1)} + \text{row 3} \end{array}$$

$$\left[\begin{array}{rrr|r} 1 & 0.5 & 3 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -9.25 & 1 \end{array}\right] \begin{array}{l} \\ \\ -\frac{1}{4}\text{(row 2)} + \text{row 3} \end{array}$$

$$\left[\begin{array}{rrr|r} 2 & 1 & 6 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -37 & 4 \end{array}\right] \begin{array}{l} 2\text{ (row 1)} \\ \\ 4\text{ (row 3)} \end{array}$$

4. a. Answers may vary. For example:

$$\left[\begin{array}{rrr|r} -1 & 0 & 1 & 2 \\ 0 & -1 & 2 & 0 \\ \frac{1}{2} & -\frac{3}{4} & -2 & \frac{1}{3} \end{array}\right]$$

$$\left[\begin{array}{rrr|r} 1 & 0 & -1 & -2 \\ 0 & -1 & 2 & 0 \\ 6 & -9 & -24 & 4 \end{array}\right] \begin{array}{l} -1\text{ (row 1)} \\ \\ 12\text{ (row 3)} \end{array}$$

$$\left[\begin{array}{rrr|r} 1 & 0 & -1 & -2 \\ 0 & -1 & 2 & 0 \\ 0 & -9 & -18 & 16 \end{array}\right] \begin{array}{l} \\ \\ -6\text{ (row 1)} + \text{row 3} \end{array}$$

$$\left[\begin{array}{rrr|r} 1 & 0 & -1 & -2 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -36 & 16 \end{array}\right] \begin{array}{l} \\ \\ -9\text{ (row 2)} + \text{row 3} \end{array}$$

b. $$\left[\begin{array}{rrr|r} 1 & 0 & -1 & -2 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -36 & 16 \end{array}\right]$$

$$-36z = 16 \text{ or } z = -\frac{4}{9}$$
$$-y + 2z = 0$$
$$-y + 2\left(-\frac{4}{9}\right) = 0 \text{ or } y = -\frac{8}{9}$$
$$x - z = -2$$
$$x - \left(-\frac{4}{9}\right) = -2 \text{ or } x = -\frac{22}{9}$$
$$x = -\frac{22}{9}, y = -\frac{8}{9}, z = -\frac{4}{9}$$

5. a. Each row of an augmented matrix corresponds to an equation in a system. The numbers on the left of the vertical line correspond to the coefficients of the unknowns, while the number to the right of the vertical line correspond to numerical answer to the equation.

$$\left[\begin{array}{cc|c} 1 & -2 & -1 \\ 2 & -3 & 1 \\ 2 & -1 & 0 \end{array}\right]$$

$$x - 2y = -1$$
$$2x - 3y = 1$$
$$2x - y = 0$$

b. $\left[\begin{array}{ccc|c} -2 & 0 & -1 & 0 \\ 1 & -2 & 0 & 4 \\ 0 & 1 & 2 & -3 \end{array}\right]$

$$-2x - z = 0$$
$$x - 2y = 4$$
$$y + 2z = -3$$

c. $\left[\begin{array}{ccc|c} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -2 \\ 0 & 1 & 1 & 0 \end{array}\right]$

$$-z = 0$$
$$x = -2$$
$$y + z = 0$$

6. a. This system can be solved using back substitution.

$$\left[\begin{array}{cc|c} -2 & 1 & 6 \\ 0 & -5 & 15 \end{array}\right]$$

$$-5y = 15 \text{ or } y = -3$$
$$-2x + y = 6$$
$$-2x - 3 = 6 \text{ or } x = -\frac{9}{2}$$
$$x = -\frac{9}{2}, y = -3$$

b. This system can be solved using back substitution.

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 11 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 6 & -36 \end{array}\right]$$

$$6z = -36 \text{ or } z = -6$$
$$2y + 3z = 0$$
$$2y + 3(-6) = 0 \text{ or } y = 9$$
$$2x - y + z = 11$$
$$2x - (9) + (-6) = 11 \text{ or } x = 13$$
$$x = 13, y = 9, z = -6$$

c. A solution does not exist to this system, because the last row has no variables, but is still equal to a non-zero number, which is not possible.

$$\left[\begin{array}{ccc|c} -1 & 3 & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & -13 \end{array}\right]$$

d. This system can be solved using back substitution.

$$\left[\begin{array}{ccc|c} 4 & -1 & -1 & 0 \\ 0 & -1 & 0 & 4 \\ 0 & 0 & -1 & 5 \end{array}\right]$$

$$-z = 5 \text{ or } z = -5$$
$$-y = 4 \text{ or } y = -4$$
$$4x - y - z = 0$$
$$4x - (-4) - (-5) = 0 \text{ or } x = -\frac{9}{4}$$
$$x = -\frac{9}{4}, y = -4, z = -5$$

e. $\left[\begin{array}{ccc|c} 1 & -1 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x - y + 3z = 2$

Set $z = t$ and $y = s$,

$x - (s) + 3(t) = 2$ or $x = 2 - 3t + s$

$x = 2 - 3t + s, y = s, z = t, s, t \in \mathbf{R}$

f. This system can be solved using back substitution.

$$\left[\begin{array}{ccc|c} -1 & 0 & 0 & -4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & -1 & 2 \end{array}\right]$$

$$-z = 2 \text{ or } z = -2$$
$$y + 2z = 4$$
$$y + 2(-2) = 4 \text{ or } y = 8$$
$$-x = -4 \text{ or } x = 4$$
$$x = 4, y = 8, z = -2$$

7. a. This matrix is in row-echelon form, because it satisfies both properties of a matrix in row-echelon form.

1. All rows that consist entirely of zeros must be written at the bottom of the matrix.

2. In any two successive rows not consisting entirely of zeros, the first nonzero number in the lower row must occur further to the right that the first nonzero number in the row directly above.

b. A solution does not exist to this system, because the second row has no variables, but is still equal to a nonzero number, which is not possible.

c. Answers may vary. For example:

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 3 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 3 \\ -2 & 2 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{ row 2 + 2(row 1)}$$

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 3 \\ -2 & 2 & 2 & 3 \\ -1 & 1 & 1 & 3 \end{array}\right] \text{ row 1 + row 3}$$

8. a. This matrix is not in row-echelon form. Answers may vary.

$$\left[\begin{array}{ccc|c} -1 & 0 & 1 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} -1 & 0 & 1 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \end{array}\right] \text{ row 3 − row 2}$$

b. This matrix is not in row-echelon form. Answers may vary.

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -3 \\ 3 & 1 & -4 & 2 \\ 0 & 0 & 3 & 6 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -3 \\ 0 & 1 & -10 & 11 \\ 0 & 0 & 3 & 6 \end{array}\right] \text{ row 2 − 3(row 1)}$$

c. This matrix is not in row-echelon form. Answers may vary. Switch row 2 and row 3 to obtain row-echelon form.

$$\left[\begin{array}{ccc|c} -1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -6 \end{array}\right] \text{ Interchange row 2 \& row 3}$$

$$\left[\begin{array}{ccc|c} -1 & 2 & 1 & 0 \\ 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

d. This matrix is in row-echelon form, because it satisfies both conditions of a matrix in row-echelon form.

1. All rows that consist entirely of zeros must be written at the bottom of the matrix.

2. If any two successive rows not consisting entirely of zeros, the first nonzero number in the lower row must occur further to the right that the first nonzero number in the row directly above.

9. a. i. $\left[\begin{array}{ccc|c} -1 & 0 & 1 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \end{array}\right]$

$2z = 1$ or $z = \frac{1}{2}$

$y = 0$

Using back substitution,

$-x + z = 3$

$-x + \frac{1}{2} = 3$ or $x = -\frac{5}{2}$

$x = -\frac{5}{2}, y = 0, z = \frac{1}{2}$

ii. $\left[\begin{array}{ccc|c} 1 & 0 & 2 & -3 \\ 0 & 1 & -10 & 11 \\ 0 & 0 & 3 & 6 \end{array}\right]$

$3z = 6$ or $z = 2$

Using back substitution,

$y - 10z = 11$

$y - 10(2) = 11$ or $y = 31$

$x + 2z = -3$

$x + 2(2) = -3$ or $x = -7$

$x = -7, y = 31, z = 2$

iii. $\left[\begin{array}{ccc|c} -1 & 2 & 1 & 0 \\ 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$z = -6$

$-x + 2y + z = 0$

Set $y = t$ and $z = -6$,

$-x + 2(t) + (-6) = 0$

$x = 2t - 6$

$x = 2t - 6, y = t, z = -6, t \in \mathbf{R}$

iv. $\left[\begin{array}{ccc|c} 1 & -4 & 1 & 0 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$y + 2z = -3$

Set $z = t$,

$y + 2(t) = -3$ or $y = -3 - 2t$

$x - 4y + z = 0$

$x - 4(-3 - 2t) + t = 0$

$x = -12 - 9t$

$x = -12 - 9t, y = -3 - 2t, z = t, t \in \mathbf{R}$

b. i. The solution is the point at which the three planes meet.

ii. The solution is the point at which the three planes meet.

iii. The solution is the line at which the three planes meet. There is one parameter t.

iv. The solution is the line at which the three planes meet. There is one parameter t.

10. a. ① $-x + y + z = 9$

② $x - 2y + z = 15$

③ $2x - y - z = -12$

$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 9 \\ 1 & -2 & 1 & 15 \\ 2 & -1 & -1 & -12 \end{array}\right]$

$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 9 \\ 0 & -1 & 2 & 24 \\ 0 & 1 & 1 & 6 \end{array}\right]$ $\begin{array}{l} \\ \text{row 1 + row 2} \\ \text{row 3 + 2(row 1)} \end{array}$

$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 9 \\ 0 & -1 & 2 & 24 \\ 0 & 0 & 3 & 30 \end{array}\right]$ $\begin{array}{l} \\ \\ \text{row 2 + row 3} \end{array}$

$3z = 30$ or $z = 10$,

$-y + 2z = 24$

$-y + 2(10) = 24$ or $y = -4$

$-x + y + z = 9$

$-x + (-4) + (10) = 9$ or $x = -3$

$x = -3, y = -4, z = 10$

These three planes meet at the point $(-3, -4, 10)$.

b. ① $x + y + z = 0$

② $2x + 3y + z = 0$

③ $-3x - 2y - 4z = 0$

$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 3 & 1 & 0 \\ -3 & -2 & -4 & 0 \end{array}\right]$

$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$ $\begin{array}{l} \\ \text{row 2} - 2\text{(row 1)} \\ \text{row 3 + 3(row 1)} \end{array}$

$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\begin{array}{l} \\ \\ \text{row 2 + row 3} \end{array}$

$y - z = 0$

Set $z = t$,

$y = t$

$x + y + z = 0$

$x + t + t = 0$

$x = -2t$

$x = -2t, y = t, z = t, t \in \mathbf{R}$

These three planes meet at this line.

c. ① $x - y + 3z = -1$

② $5x + y - 3z = -5$

③ $2x + y - 3z = -2$

$\left[\begin{array}{ccc|c} 1 & -1 & 3 & -1 \\ 5 & 1 & -3 & -5 \\ 2 & 1 & -3 & -2 \end{array}\right]$

$\left[\begin{array}{ccc|c} 1 & -1 & 3 & -1 \\ 0 & 6 & -18 & 0 \\ 0 & 3 & -9 & 0 \end{array}\right]$ $\begin{array}{l} \\ \text{row 2} - 5\text{(row 1)} \\ \text{row 3} - 2\text{(row 1)} \end{array}$

$\left[\begin{array}{ccc|c} 1 & -1 & 3 & -1 \\ 0 & 6 & -18 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\begin{array}{l} \\ \\ \text{row 3} - \frac{1}{2}\text{(row 2)} \end{array}$

$6y - 18z = 0$

Set $z = t$,

$6y - 18t = 0$ or $y = 3t$

$x - y + 3z = -1$

Set $z = t$ and $y = 3t$,

$x - (3t) + 3t = -1$ or $x = -1$

$x = -1, y = 3t, z = t, t \in \mathbf{R}$

These three planes meet at this line.

d. ① $x + 3y + 4z = 4$

② $-x + 3y + 8z = -4$

③ $x - 3y - 4z = -4$

$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 4 \\ -1 & 3 & 8 & -4 \\ 1 & -3 & -4 & -4 \end{array}\right]$

$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 4 \\ 0 & 6 & 12 & 0 \\ 0 & -6 & -8 & -8 \end{array}\right]$ $\begin{array}{l} \\ \text{row 2 + row 1} \\ \text{row 3} - \text{row 1} \end{array}$

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 4 \\ 0 & 6 & 12 & 0 \\ 0 & 0 & 4 & -8 \end{array}\right] \text{ row 3 + row 2}$$

$4z = -8$ or $z = -2$

$6y + 12z = 0$

$6y + 12(-2) = 0$

$y = 4$

$x + 3y + 4z = 4$

$x + 3(4) + 4(-2) = 4$

$x = 0$

$x = 0, y = 4, z = -2$

The three planes meet at the point $(0, 4, -2)$.

e. ① $2x + y + z = 1$

② $4x + 2y + 2z = 2$

③ $-2x + y + z = 3$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 4 & 2 & 2 & 2 \\ -2 & 1 & 1 & 3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 4 \end{array}\right] \begin{array}{l} \\ \text{row 2} - 2(\text{row 1}) \\ \text{row 3 + row 1} \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{ Interchange row 2 \& row 3}$$

$2y + 2z = 4$

Set $z = t$,

$2y + 2t = 4$ or $y = 2 - t$

$2x + y + z = 1$

$2x + (2 - t) + t = 1$

$x = -\frac{1}{2}$

$x = -\frac{1}{2}, y = 2 - t, z = t, t \in \mathbf{R}$

The three planes meet at this line.

f. ① $x - y = -500$

② $2y - z = 3500$

③ $x - z = 2000$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & -500 \\ 0 & 2 & -1 & 3500 \\ 1 & 0 & -1 & 2000 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & -500 \\ 0 & 2 & -1 & 3500 \\ 0 & 1 & -1 & 2500 \end{array}\right] \text{ row 3 − row 1}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & -500 \\ 0 & 2 & -1 & 3500 \\ 0 & 0 & -0.5 & 750 \end{array}\right] \text{ row 3} - \frac{1}{2}(\text{row 2})$$

$-0.5z = 750$ or $z = -1500$

$2y - z = 3500$

$2y - (-1500) = 3500$ or $y = 1000$

$x - y = -500$

$x - (1000) = -500$ or $x = 500$

$x = 500, y = 1000, z = -1500$

The three planes meet at the point $(500, 1000, -1500)$.

11. $a = x + 2y - z$

$b = x - y + 2z$

$c = 3x + 3y + z$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & a \\ 1 & -1 & 2 & b \\ 3 & 3 & 1 & c \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & a \\ 0 & -3 & 3 & -a + b \\ 0 & -3 & 4 & -3a + c \end{array}\right] \begin{array}{l} \\ \text{row 2 − row 1} \\ \text{row 3} - 3(\text{row 1}) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & a \\ 0 & -3 & 3 & -a + b \\ 0 & 0 & 1 & -2a - b + c \end{array}\right] \text{ row 3 − row 2}$$

$z = -2a - b + c$

$-3y + 3z = -a + b$

$-3y + 3(-2a - b + c) = -a + b$

$y = \frac{-5a - 4b + 3c}{3}$

$x + 2y - z = a$

$x + 2\left(\frac{-5a - 4b + 3c}{3}\right) - (-2a - b + c) = a$

$x = \frac{7a + 5b - 3c}{3}$

$x = \frac{7a + 5b - 3c}{3}, y = \frac{-5a - 4b + 3c}{3},$

$z = -2a - b + c$

12. First write an equation corresponding to each point using the given equation $y = ax^2 + bx + c$. Then create a matrix corresponding to these three equations, and solve for a, b, and c.

$A(-1, -7): -7 = a(-1)^2 + b(-1) + c$

$-7 = a - b + c$

$B(2, 20): 20 = a(2)^2 + b(2) + c$

$20 = 4a + 2b + c$

$C(-3, -5): -5 = a(-3)^2 + b(-3) + c$

$-5 = 9a - 3b + c$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -7 \\ 4 & 2 & 1 & 20 \\ 9 & -3 & 1 & -5 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -7 \\ 0 & 6 & -3 & 48 \\ 0 & 6 & -8 & 58 \end{array}\right] \begin{array}{l} \\ \text{row 2} - 4(\text{row 1}) \\ \text{row 3} - 9(\text{row 1}) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -7 \\ 0 & 6 & -3 & 48 \\ 0 & 0 & -5 & 10 \end{array}\right] \begin{array}{l} \\ \\ \text{row 3} - \text{row 2} \end{array}$$

$-5c = 10$ or $c = -2$

$6b - 3c = 48$

$6b - 3(-2) = 48$ or $b = 7$

$a - b + c = -7$ or $a = 2$

$a = 2, b = 7, c = -2$

$y = 2x^2 + 7x - 2$

13. First change the variables in the equations to easier variables to work with, and then use matrices to solve for the unknowns. Once you have figured out the unknowns you created, use back substitution to solve for *p, q,* and *r.*

① $pq - 2\sqrt{q} + 3rq = 8$
② $2pq - \sqrt{q} + 2rq = 7$
③ $-pq + \sqrt{q} + 2rq = 4$

Let $x = pq$, $y = \sqrt{q}$, and $z = rq$.

① $x - 2y + 3z = 8$
② $2x - y + 2z = 7$
③ $-x + y + 2z = 4$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 8 \\ 2 & -1 & 2 & 7 \\ -1 & 1 & 2 & 4 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 8 \\ 0 & 3 & -4 & -9 \\ 0 & -1 & 5 & 12 \end{array}\right] \begin{array}{l} \\ \text{row 2} - 2(\text{row 1}) \\ \text{row 3} + \text{row 1} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 8 \\ 0 & 3 & -4 & -9 \\ 0 & 0 & \frac{11}{3} & 9 \end{array}\right] \begin{array}{l} \\ \\ \text{row 3} + \frac{1}{3}(\text{row 2}) \end{array}$$

$\frac{11}{3}z = 9$ or $z = \frac{27}{11}$

$3y - 4z = -9$

$3y - 4\left(\frac{27}{11}\right) = -9$

$y = \frac{3}{11}$

$x - 2y + 3z = 8$

$x - 2\left(\frac{3}{11}\right) + 3\left(\frac{27}{11}\right) = 8$

$x = \frac{13}{11}$

$x = \frac{13}{11}, y = \frac{3}{11}, z = \frac{27}{11}$

$y = \frac{3}{11} = \sqrt{q}$ or $q = \frac{9}{121}$

$x = \frac{13}{11} = pq = p\left(\frac{9}{121}\right)$

$p = \frac{143}{9}$

$z = \frac{27}{11} = rq = r\left(\frac{9}{121}\right)$

$r = 33$

$p = \frac{143}{9}, q = \frac{9}{121}, r = 33$

14. $\left[\begin{array}{ccc|c} a & 1 & 1 & a \\ 1 & a & 1 & a \\ 1 & 1 & a & a \end{array}\right]$

$$\left[\begin{array}{ccc|c} 1 & 1 & a & a \\ 1 & a & 1 & a \\ a & 1 & 1 & a \end{array}\right] \text{ Interchange row 1 \& row 3}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & a & a \\ 0 & a-1 & 1-a & 0 \\ 0 & 1-a & 1-a^2 & a-a^2 \end{array}\right] \begin{array}{l} \\ \text{row 2} - \text{row 1} \\ \text{row 3} - a(\text{row 1}) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & a & a \\ 0 & a-1 & 1-a & 0 \\ 0 & 0 & (-a-2)(a-1) & a-a^2 \end{array}\right]$$

row 2 + row 3

Analyzing this matrix, you can tell that when $a = 1$, the matrix has a row of all zeros, which means that this matrix has an infinite number of solutions. If you substitute in $a = -2$, the matrix has a row of variables with the coefficient zero, but a non zero number, which means this system has no solution. Any other number that is substituted for a gives a unique solution.

a. $a = -2$

b. $a = 1$

c. $a \neq -2$ or $a \neq 1$

Gauss-Jordan Method for Solving Systems of Equations, pp. 594–595

1. a. $\left[\begin{array}{rr|r} -1 & 3 & 1 \\ 0 & -1 & 2 \end{array}\right]$

$\left[\begin{array}{rr|r} -1 & 0 & 7 \\ 0 & -1 & 2 \end{array}\right]$ row 1 + 3 (row 2)

$\left[\begin{array}{rr|r} 1 & 0 & -7 \\ 0 & 1 & -2 \end{array}\right]$ $\begin{array}{l} -1 \text{ (row 1)} \\ -1 \text{ (row 2)} \end{array}$

b. $\left[\begin{array}{rrr|r} 1 & 0 & 2 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -1 & 0 \end{array}\right]$

$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & 0 \end{array}\right]$ $\begin{array}{l} \text{row 1 + 2 (row 3)} \\ \text{row 2 + 2 (row 3)} \\ \end{array}$

$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right]$ -1 (row 3)

c. $\left[\begin{array}{rrr|r} 1 & 1 & 4 & 2 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$

$\left[\begin{array}{rrr|r} 1 & 0 & 6 & 1 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$ row 1 + row 2

$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$ $\begin{array}{l} \text{row 1} - \text{6 (row 3)} \\ \text{row 2} - \text{2 (row 3)} \\ \end{array}$

$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$ -1 (row 2)

d. $\left[\begin{array}{rrr|r} 0 & 0 & -1 & 4 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & -1 \end{array}\right]$

$\left[\begin{array}{rrr|r} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 4 \end{array}\right]$ Rearrange rows

$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -4 \end{array}\right]$ $\begin{array}{l} \text{row 1 + 2 (row 3)} \\ \\ -1 \text{ (row 3)} \end{array}$

2. The solution to each reduced row-echelon matrix is the numbers corresponding to each leading 1.
a. $(-7, -2)$
b. $(3, 2, 0)$
c. $(1, 1, 0)$
d. $(8, -1, -4)$

3. a. ① $x - y + z = 0$
② $x + 2y - z = 8$
③ $2x - 2y + z = -11$

$\left[\begin{array}{rrr|r} 1 & -1 & 1 & 0 \\ 1 & 2 & -1 & 8 \\ 2 & -2 & 1 & -11 \end{array}\right]$

$\left[\begin{array}{rrr|r} 1 & -1 & 1 & 0 \\ 0 & 3 & -2 & 8 \\ 0 & 0 & -1 & -11 \end{array}\right]$ $\begin{array}{l} \\ \text{row 2} - \text{row 1} \\ \text{row 3} - \text{2 (row 1)} \end{array}$

$\left[\begin{array}{rrr|r} 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{8}{3} \\ 0 & 0 & -1 & -11 \end{array}\right]$ $\frac{1}{3}$ (row 2)

$\left[\begin{array}{rrr|r} 1 & 0 & \frac{1}{3} & \frac{8}{3} \\ 0 & 1 & -\frac{2}{3} & \frac{8}{3} \\ 0 & 0 & -1 & -11 \end{array}\right]$ row 1 + row 2

$\left[\begin{array}{rrr|r} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 11 \end{array}\right]$ $\begin{array}{l} \text{row 1} + \frac{1}{3} \text{ row 3} \\ \text{row 2} - \frac{2}{3} \text{ row 3} \\ -1 \text{ (row 3)} \end{array}$

The solution to this system of equations is $x = -1$, $y = 10$, and $z = 11$.

b. ① $3x - 2y + z = 6$
② $x - 3y - 2z = -26$
③ $-x + y + z = 9$

$\left[\begin{array}{rrr|r} 3 & -2 & 1 & 6 \\ 1 & -3 & -2 & -26 \\ -1 & 1 & 1 & 9 \end{array}\right]$

$\left[\begin{array}{rrr|r} 1 & -3 & -2 & -26 \\ 3 & -2 & 1 & 6 \\ -1 & 1 & 1 & 9 \end{array}\right]$ Interchange rows 1 & 2

$\left[\begin{array}{rrr|r} 1 & -3 & -2 & -26 \\ 0 & 7 & 7 & 84 \\ 0 & -2 & -1 & -17 \end{array}\right]$ $\begin{array}{l} \text{row 2} - \text{3(row 1)} \\ \text{row 3 + row 1} \\ \end{array}$

$$\left[\begin{array}{ccc|c} 1 & -3 & -2 & -26 \\ 0 & 1 & 1 & 12 \\ 0 & -2 & -1 & -17 \end{array}\right] \frac{1}{7}(\text{row } 2)$$

$$\left[\begin{array}{ccc|c} 1 & -3 & -2 & -26 \\ 0 & 1 & 1 & 12 \\ 0 & 0 & 1 & 7 \end{array}\right] \begin{array}{l} \\ \\ \text{row } 3 + 2(\text{row } 2) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 10 \\ 0 & 1 & 1 & 12 \\ 0 & 0 & 1 & 7 \end{array}\right] \begin{array}{l} \text{row } 1 + 3\ (\text{row } 2) \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 7 \end{array}\right] \begin{array}{l} \text{row } 1 - \text{ row } 3 \\ \text{row } 2 - \text{ row } 3 \\ \\ \end{array}$$

The solution to this system of equations is $x = 3$, $y = 5$, and $z = 7$.

c. ① $2x + 2y + 5z = -14$
② $-x + z = -5$
③ $y - z = 6$

$$\left[\begin{array}{ccc|c} 2 & 2 & 5 & -14 \\ -1 & 0 & 1 & -5 \\ 0 & 1 & -1 & 6 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} -1 & 0 & 1 & -5 \\ 2 & 2 & 5 & -14 \\ 0 & 1 & -1 & 6 \end{array}\right] \begin{array}{l} \text{Interchange rows 1 \& 2} \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} -1 & 0 & 1 & -5 \\ 0 & 2 & 7 & -24 \\ 0 & 1 & -1 & 6 \end{array}\right] \text{row } 2 + 2(\text{row } 1)$$

$$\left[\begin{array}{ccc|c} -1 & 0 & 1 & -5 \\ 0 & 2 & 7 & -24 \\ 0 & 0 & -\frac{9}{2} & 18 \end{array}\right] \begin{array}{l} \\ \\ \text{row } 3 - \frac{1}{2}(\text{row } 2) \end{array}$$

$$\left[\begin{array}{ccc|c} -1 & 0 & 1 & -5 \\ 0 & 2 & 7 & -24 \\ 0 & 0 & 1 & -4 \end{array}\right] \begin{array}{l} \\ \\ -\frac{2}{9}(\text{row } 3) \end{array}$$

$$\left[\begin{array}{ccc|c} -1 & 0 & 0 & -1 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & 1 & -4 \end{array}\right] \begin{array}{l} \text{row } 1 - \text{row } 3 \\ \text{row } 2 - 7(\text{row } 3) \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -4 \end{array}\right] \begin{array}{l} -1\ (\text{row } 1) \\ \frac{1}{2}(\text{row } 2) \\ \\ \end{array}$$

The solution to this system of equations is $x = 1$, $y = 2$, and $z = -4$.

d. ① $x - y - 3z = 3$
② $2x + 2y + z = -1$
③ $-x - y + z = -1$

$$\left[\begin{array}{ccc|c} 1 & -1 & -3 & 3 \\ 2 & 2 & 1 & -1 \\ -1 & -1 & 1 & -1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -3 & 3 \\ 0 & 4 & 7 & -7 \\ 0 & -2 & -2 & 2 \end{array}\right] \begin{array}{l} \\ \text{row } 2 - 2(\text{row } 1) \\ \text{row } 3 + \text{row } 1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -3 & 3 \\ 0 & 4 & 7 & -7 \\ 0 & 0 & \frac{3}{2} & -\frac{3}{2} \end{array}\right] \begin{array}{l} \\ \\ \text{row } 3 + \frac{1}{2}(\text{row } 2) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -3 & 3 \\ 0 & 4 & 7 & -7 \\ 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{l} \\ \\ \frac{2}{3}(\text{row } 3) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -3 & 3 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] \text{row } 2 - 7(\text{row } 3)$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -3 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] \frac{1}{4}(\text{row } 2)$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{l} \text{row } 1 + \text{row } 2 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{l} \text{row } 1 + 3(\text{row } 3) \\ \\ \\ \end{array}$$

The solution to this system of equations is $x = 0$, $y = 0$, and $z = -1$.

e. ① $\frac{1}{2}x + 9y - z = 1$
② $x - 6y + z = -6$
③ $2x + 3y - z = -7$

$$\left[\begin{array}{ccc|c} \frac{1}{2} & 9 & -1 & 1 \\ 1 & -6 & 1 & -6 \\ 2 & 3 & -1 & -7 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -6 & 1 & -6 \\ \frac{1}{2} & 9 & -1 & 1 \\ 2 & 3 & -1 & -7 \end{array}\right] \begin{array}{l} \text{Interchange rows 1 \& 2} \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -6 & 1 & -6 \\ 1 & 18 & -2 & 2 \\ 2 & 3 & -1 & -7 \end{array}\right] \begin{array}{l} \\ 2(\text{row } 2) \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -6 & 1 & -6 \\ 0 & 24 & -3 & 8 \\ 0 & 15 & -3 & 5 \end{array}\right] \begin{array}{l} \\ \text{row } 2 - \text{row } 1 \\ \text{row } 3 - 2(\text{row } 1) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{4} & -4 \\ 0 & 24 & -3 & 8 \\ 0 & 0 & -\frac{9}{8} & 0 \end{array}\right] \begin{array}{l} \text{row } 1 + \frac{1}{4}(\text{row } 2) \\ \\ \text{row } 3 - \frac{15}{24}(\text{row } 2) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{4} & -4 \\ 0 & -24 & -3 & 8 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} \\ \\ -\frac{8}{9}(\text{row } 3) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 2 & 3 & \frac{2}{3} \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} \text{row } 1 - \frac{1}{4}(\text{row } 3) \\ \text{row } 2 + 3(\text{row } 3) \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 2 & 0 & \frac{2}{3} \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} \\ \text{row } 2 - 3(\text{row } 3) \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} \\ \frac{1}{2}(\text{row } 2) \\ \\ \end{array}$$

The solution to this system of equations is $x = -4$, $y = \frac{1}{3}$, and $z = 0$.

f. ① $2x + 3y + 6z = 3$
② $x - y - z = 0$
③ $4x + 3y - 6z = 2$

$$\left[\begin{array}{ccc|c} 2 & 3 & 6 & 3 \\ 1 & -1 & -1 & 0 \\ 4 & 3 & -6 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 2 & 3 & 6 & 3 \\ 4 & 3 & -6 & 2 \end{array}\right] \begin{array}{l} \text{Interchange row 1 \& row 2} \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 5 & 8 & 3 \\ 0 & 7 & -2 & 2 \end{array}\right] \begin{array}{l} \\ \text{row } 2 - 2(\text{row } 1) \\ \text{row } 3 - 4(\text{row } 1) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 5 & 8 & 3 \\ 0 & 0 & -\frac{66}{5} & -\frac{11}{5} \end{array}\right] \begin{array}{l} \\ \\ \text{row } 3 - \frac{7}{5}(\text{row } 2) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{3}{5} & \frac{3}{5} \\ 0 & 5 & 8 & 3 \\ 0 & 0 & 1 & \frac{1}{6} \end{array}\right] \begin{array}{l} \text{row } 1 + \frac{1}{5}(\text{row } 2) \\ \\ -\frac{5}{66}(\text{row } 3) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 5 & 0 & \frac{5}{3} \\ 0 & 0 & 1 & \frac{1}{6} \end{array}\right] \begin{array}{l} \text{row } 1 - \frac{3}{5}(\text{row } 3) \\ \text{row } 2 - 8(\text{row } 3) \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{6} \end{array}\right] \begin{array}{l} \\ \frac{1}{5}(\text{row } 2) \\ \\ \end{array}$$

The solution to this system of equations is $x = \frac{1}{2}$, $y = \frac{1}{3}$, and $z = \frac{1}{6}$.

4. a. ① $2x + y - z = -6$
② $x - y + 2z = 9$
③ $-x + y + z = 9$

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & -6 \\ 1 & -1 & 2 & 9 \\ -1 & 1 & 1 & 9 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 9 \\ 2 & 1 & -1 & -6 \\ -1 & 1 & 1 & 9 \end{array}\right] \begin{array}{l} \text{Interchange row 1 \& row 2} \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 9 \\ 0 & 3 & -5 & -24 \\ 0 & 0 & 3 & 18 \end{array}\right] \begin{array}{l} \\ \text{row } 2 - 2(\text{row } 1) \\ \text{row } 3 + \text{row } 1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{3} & 1 \\ 0 & 3 & -5 & -24 \\ 0 & 0 & 1 & 6 \end{array}\right] \begin{array}{l} \text{row } 1 + \frac{1}{3}(\text{row } 2) \\ \\ \frac{1}{3}(\text{row } 3) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 3 & 0 & 6 \\ 0 & 0 & 1 & 6 \end{array}\right] \begin{array}{l} \text{row } 1 - \frac{1}{3}(\text{row } 3) \\ \text{row } 2 + 5(\text{row } 3) \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 6 \end{array}\right] \quad \frac{1}{3}\text{(row 2)}$$

The solution to this system of equations is $x = -1$, $y = 2$, and $z = 6$.

b. ① $x - y + z = -30$
② $-2x + y + 6z = 90$
③ $2x - y - z = -20$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -30 \\ -2 & 1 & 6 & 90 \\ 2 & -1 & -1 & -20 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -30 \\ -2 & 1 & 6 & 90 \\ 0 & 0 & 5 & 70 \end{array}\right] \quad \text{row 2 + row 3}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -30 \\ 0 & -1 & 8 & 30 \\ 0 & 0 & 1 & 14 \end{array}\right] \quad \begin{array}{l} \text{row 2 + 2(row 1)} \\ \frac{1}{5}\text{(row 3)} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -7 & -60 \\ 0 & -1 & 8 & 30 \\ 0 & 0 & 1 & 14 \end{array}\right] \quad \text{row 1 − 1(row 2)}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 38 \\ 0 & -1 & 0 & -82 \\ 0 & 0 & 1 & 14 \end{array}\right] \quad \begin{array}{l} \text{row 1 + 7(row 3)} \\ \text{row 2 − 8(row 3)} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 38 \\ 0 & 1 & 0 & 82 \\ 0 & 0 & 1 & 14 \end{array}\right] \quad -1\text{(row 2)}$$

The solution to this system of equations is $x = 38$, $y = 82$, and $z = 14$.

5. ① $x + y + z = -1$
② $x - y + z = 2$
③ $3x - y + 3z = k$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 2 \\ 3 & -1 & 3 & k \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & -2 & 0 & 3 \\ 0 & -4 & 0 & k+3 \end{array}\right] \quad \begin{array}{l} \text{row 2 − row 1} \\ \text{row 3 − 3(row 1)} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & -2 & 0 & 3 \\ 0 & 0 & 0 & k-3 \end{array}\right] \quad \text{row 3 − 2(row 2)}$$

a. There is an infinite number of solutions for this system when $k = 3$, because this creates a zero row.

b. The system will have no solution when $k \neq 3$, $k \in \mathbf{R}$ because there will be a row of zeros equal to a nonzero number, which is not possible.

c. The system cannot have a unique solution, because the matrix cannot be put in reduced row-echelon form.

6. a. Every homogeneous system has at least one solution, because (0, 0, 0) satisfies each equation.

b. ① $2x - y + z = 0$
② $x + y + z = 0$
③ $5x - y + 3z = 0$

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 5 & -1 & 3 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & -1 & 1 & 0 \\ 5 & -1 & 3 & 0 \end{array}\right] \quad \text{Interchange row 1 and row 2}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & -6 & -2 & 0 \end{array}\right] \quad \begin{array}{l} \text{row 2 − 2(row 1)} \\ \text{row 3 − 5(row 1)} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{row 3 − 2(row 2)}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{2}{3} & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} \text{row 1} + \frac{1}{3}\text{(row 2)} \\ -\frac{1}{3}\text{(row 2)} \end{array}$$

The reduced row-echelon form shows that the intersection of these planes is a line that goes through the point (0, 0, 0). $x = -\frac{2}{3}t$, $y = -\frac{1}{3}t$, $z = t$, $t \in \mathbf{R}$

7. ① $\frac{1}{x} - \frac{2}{y} + \frac{6}{z} = \frac{5}{6}$

② $\frac{2}{x} - \frac{3}{y} + \frac{12}{z} = 2$

③ $\frac{3}{x} + \frac{6}{y} + \frac{2}{z} = \frac{23}{6}$

Let the variables be $\frac{1}{x}$, $\frac{1}{y}$, and $\frac{1}{z}$

$$\left[\begin{array}{ccc|c} 1 & -2 & 6 & \frac{5}{6} \\ 2 & -3 & 12 & 2 \\ 3 & 6 & 2 & \frac{23}{6} \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 6 & \frac{5}{6} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 12 & -16 & \frac{4}{3} \end{array}\right] \begin{array}{l} \\ \text{row } 2 - 2(\text{row } 1) \\ \text{row } 3 - 3(\text{row } 1) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 6 & \frac{3}{2} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & -16 & -\frac{8}{3} \end{array}\right] \begin{array}{l} \text{row } 1 + 2(\text{row } 2) \\ \\ \text{row } 3 - 12(\text{row } 2) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{6} \end{array}\right] \begin{array}{l} \text{row } 1 + \frac{6}{16}(\text{row } 3) \\ \\ -\frac{1}{16}(\text{row } 3) \end{array}$$

$\frac{1}{x} = \frac{1}{2}, x = 2$

$\frac{1}{y} = \frac{1}{3}, y = 3$

$\frac{1}{z} = \frac{1}{6}, z = 6$

$(2, 3, 6)$