1 Orienting Yourself: The Use of Coordinates

1.1 Introduction

A system of coordinates is a structured framework (like the grid lines on a map or graph paper) that enables us to use numbers to describe the exact physical locations of points or objects.

The idea of 'grid-based thinking' and the geometry required to define the locations of points in space—indeed has deep roots in Bhārat. The first systematic use of grids occurred thousands of years ago — on a massive urban scale—in the Sindhu-Sarasvatī Civilisation, where city streets were constructed with striking precision in North–South and East–West directions at uniform distances of about 10 metres apart. This was a coordinate system in practice: a merchant could find a shop or a warehouse by counting North–South and East–West units of distance from the city centre. Baudhāyana (c. 800 C.E.), as we have seen, later used East–West and North–South lines for his deep geometric constructions, developing the Baudhāyana–Pythagoras Theorem and thus laying the foundation of coordinate geometry.

Putting coordinates on the Earth's surface later became important for navigation. Ujjayinī was described in the ancient world—at least as early as the 4th century BCE in the early Siddhāntas—as the point marking the central longitude meridian from which all other locations were measured. The Greek mathematician Ptolemy (c. 150 BCE), building on earlier works including that of Hipparchus, later described the latitudes and longitudes of thousands of locations, including 'Ozine' (Ujjayinī). Āryabhaṭa (c. 499 CE) replaced the Greek 'chords' with 'sines', making it much easier to calculate the coordinates of a star or a city. He mapped the sky using Celestial Coordinates, measuring coordinate distances from the ecliptic (the path of the sun).

Brahmagupta (c. 628 CE) formalised the notion and use of zero and the negative numbers as algebraic entities; in modern coordinate systems, the 'origin' is zero and the 'negative axes' represent values less than zero. Without Brahmagupta's work, the four-quadrant Cartesian plane, as we will study in this chapter, would be impossible.

Brahmagupta's work was translated into Arabic (as the Sindhind), and the Ujjayinī meridian entered Arabic geography under the name 'Arin,' serving as the zero-longitude reference for early Arabic maps which also then made use of negative numbers. The influential Arab scholar Al-Bīrūnī (c. 1000 CE) travelled to India, studied the Siddhāntas, and used Indian trigonometric methods to calculate the coordinates of various cities across Asia. Al-Bīrūnī also later perfected the 'astrolabe', a handheld device that allowed sailors to find their coordinates by looking at the stars. Ömar Khayyām (c. 1100 CE), who had become an expert in the Indian decimal system and algebraic formalism, was the first mathematician to solve algebraic problems using geometry by interpreting them in terms of coordinates in the plane.

These concepts eventually reached Europe in the 12th century. The final leap occurred when following the related work of Fermat (1636 CE), René Descartes (1637 CE) formalised the fact that any point in a two-dimensional plane could be defined by simply two numbers—representing the point's distances from two perpendicular axes. Points and more complex geometric shapes could then be described precisely using algebra and equations, thus bringing the areas of geometry and algebra even closer together.

In Grades 9 and 10, you will have a chance to study this amazing coordinate system which has such a rich history in human thought and endeavour. You will be able to locate objects with pinpoint accuracy. You will also see how using coordinates enables us to visualise algebraic equations as geometric shapes, and vice versa.

We begin our study of coordinates with a story that will help you understand these new terms better.

1.2 Settling In

It is the beginning of the academic year and Reiaan is both excited and nervous. The family has just moved to a new city. He and his sister, Shalini, will be attending a new school. Today, Shalini will help him settle into the new environment. When someone is not able to see, this can be a very big challenge, but with their mother's transferable job, the siblings have done it often, and it has become easier with each move.

Shalini has just completed Grade 9 and this time, she decided to put to use what she has learnt in Coordinate Geometry in Mathematics to guide Reiaan.

Shalini wanted Reiaan to feel the directions, so she used a rectangular grid on which she had fixed pins and threads. This showed the floor of the room. Points in the sketch were marked using pins. Shalini was using a scale of 1 cm : 1 foot. She used pins to mark out various key points of the room. Points representing the corners of objects were connected with thick wool so that Reiaan could feel their positions with his fingers.

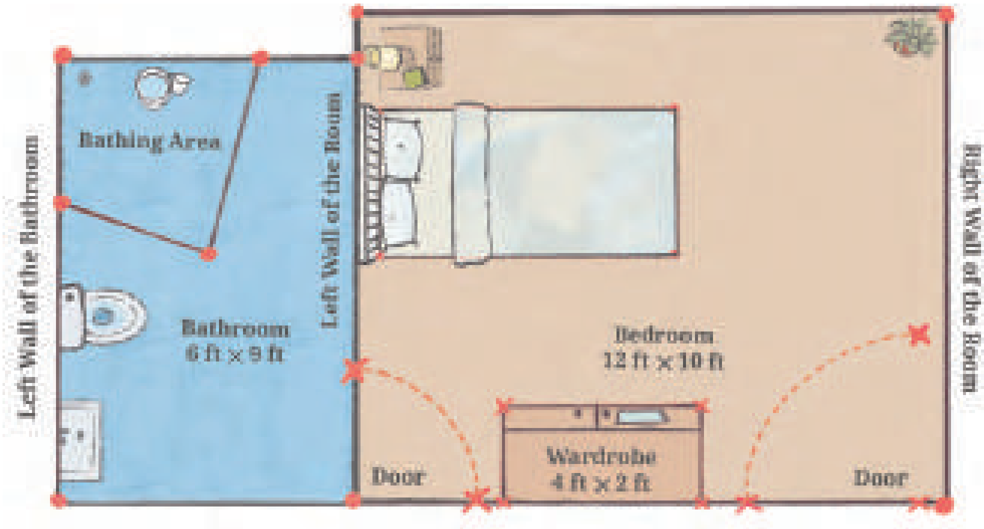


Fig. 1.1: Sketch of Reiaan's room

Let us examine Fig. 1.1 to understand the layout of the room. Notice that this only shows the map of the floor. Do you see why the position of the windows cannot be marked on this map?

1.3 The 2-D Cartesian Coordinate System

In the chapters on integers, rational numbers, and decimals in earlier grades, you studied the number line which is one-dimensional. The two-dimensional coordinate system uses two lines at right angles to each other to mark points in two-dimensional space (short form: 2-D space). For convenience, we consider one of the lines to be horizontal; it is called the x-axis. The other line is vertical; it is called the y-axis. The point of intersection of the x-axis and y-axis is called the origin O; its coordinates are (0, 0). Coordinate axes (this is the plural of 'axis') help us to locate any point in 2-D space using the point's 'coordinates'. Distances from O are marked off in equal units, on both the axes. Distances to the right of O or upwards from O are considered positive, and distances to the left of O or downwards from O are considered negative (Fig. 1.2).

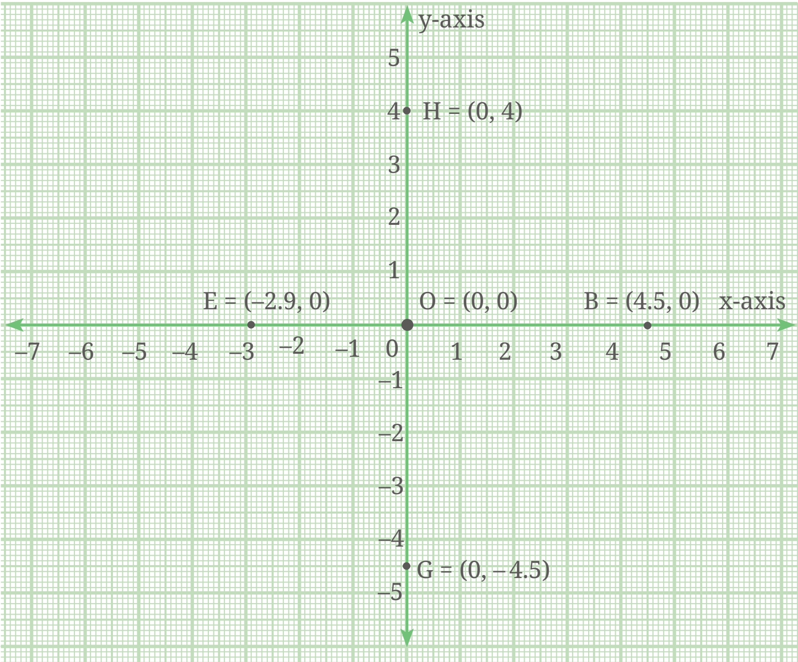


Fig. 1.2: Structure of the coordinate plane

Point B is on the x-axis and 4.5 units to the right of O, its coordinates are (4.5, 0). This fact is written as B = (4.5, 0). Point G is on the y-axis and 4.5 units downward from O, its coordinates are (0, –4.5), that is, G = (0, –4.5). Point H is also on the y-axis but 4 units above O, its coordinates are (0, 4), that is, H = (0, 4).

A point P = (x, 0) lies on the x-axis. If x is positive, then P lies to the right of O. If x is negative, P lies to the left of O. A point P = (0, y) lies on the y-axis. If y is positive, P lies above O; if y is negative, P lies below O.

While writing the coordinates of a point, it is often convenient to drop the '=' sign and write P = (x, y) simply as P (x, y). This is especially true while marking points on a graph.

Exercise Set 1.1

Fig. 1.3 shows Reiaan's room with points OABC marking its corners. The x- and y-axes are marked in the figure. Point O is the origin.

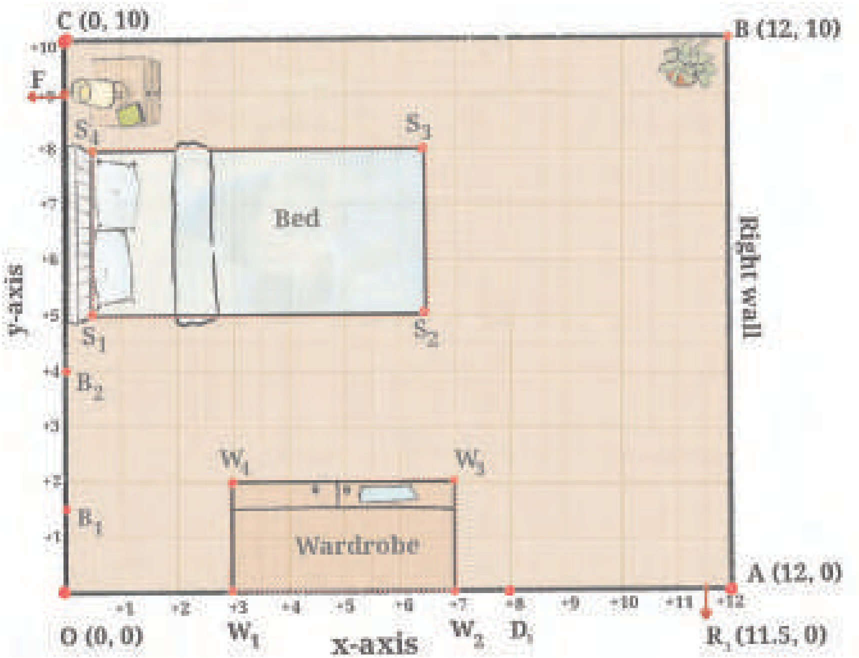


Fig. 1.3

Referring to Fig. 1.3, answer the following questions:

(i) If D_1R_1 represents the door to Reiaan's room, how far is the door from the left wall (the y-axis) of the room? How far is the door from the x-axis?

(ii) What are the coordinates of D_1?

(iii) If R_1 is the point (11.5, 0), how wide is the door? Do you think this is a comfortable width for the room door? If a person in a wheelchair wants to enter the room, will he/she be able to do so easily?

(iv) If B_1 (0, 1.5) and B_2 (0, 4) represent the ends of the bathroom door, is the bathroom door narrower or wider than the room door?

Think and Reflect

1. What are the standard widths for a room door? Look around your home and in school.
2. Are the doors in your school suitable for people in wheelchairs?

So far, we have only considered points on the two coordinate axes. What can you say about the coordinates of points that are not on either axes? The plane in which the axes are situated is called the **Cartesian plane**, the **coordinate plane** or the **xy-plane**. The axes divide the plane into four parts, called **quadrants**. They are numbered as shown in Fig. 1.4.

Using the conventions that we have stated earlier:

(i) Points in Quadrant I have both x- and y-coordinates positive.

(ii) Points in Quadrant II have negative x-coordinate and positive y-coordinate.

(iii) Points in Quadrant III have both x- and y-coordinates negative.

(iv) Points in Quadrant IV have positive x-coordinate and negative y-coordinate.

Can you now see the meaning of the coordinates of a point in 2-D space? In general, the coordinates of a point P in 2-D space are represented by (x, y). Here, x represents the perpendicular distance of P from the y-axis, measured along the x-axis, and y is the perpendicular distance of P from the x-axis, measured along the y-axis. x is the x-coordinate and y is the y-coordinate of the point (x, y).

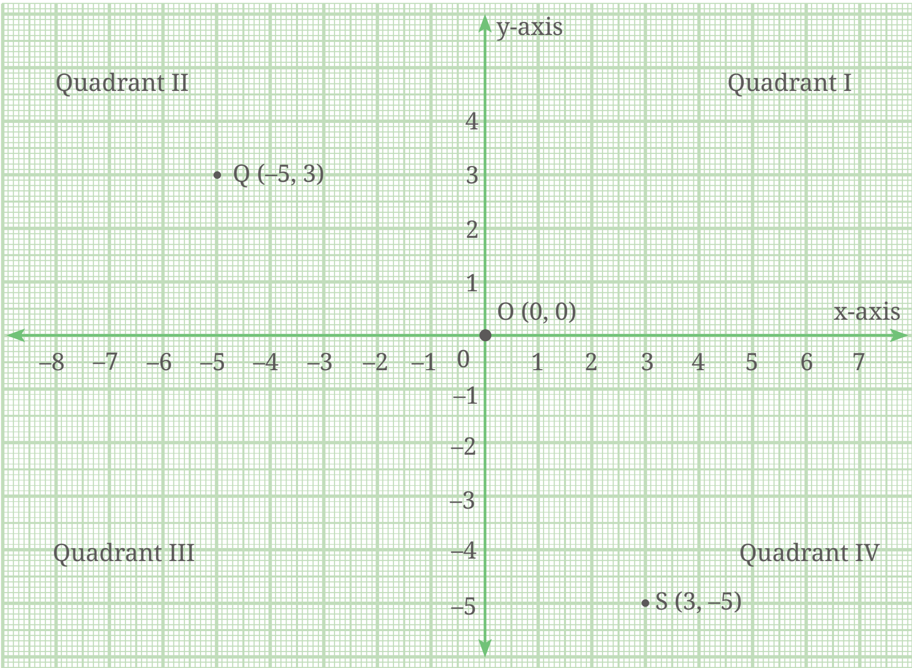


Fig. 1.4

For example, in Fig. 1.4, point S (3, –5) is in Quadrant IV, the x-coordinate is 3 units and the y-coordinate is –5 units. What about point Q (–5, 3)? It is in the second quadrant, with x-coordinate –5 and y-coordinate 3. Copy Fig. 1.4 and mark S and Q in your diagram. Mark any point P in Quadrant I and any point R in Quadrant III, and write down their coordinates.

Think and Reflect

1. What is the x-coordinate of a point on the y-axis?
2. Is there a similar generalisation for a point on the x-axis?
3. Does point Q (y, x) ever coincide with point P (x, y)? Justify your answer.
4. If $x \neq y$, then $(x, y) \neq (y, x)$; and $(x, y) = (y, x)$ if and only if $x = y$. Is this claim true?

Exercise Set 1.2

On a graph sheet, mark the x-axis and y-axis and the origin O. Mark points from (–7, 0) to (13, 0) on the x-axis and from (0, –15) to (0, 12) on the y-axis. (Use the scale 1 cm = 1 unit.) Using Fig. 1.5, answer the given questions.

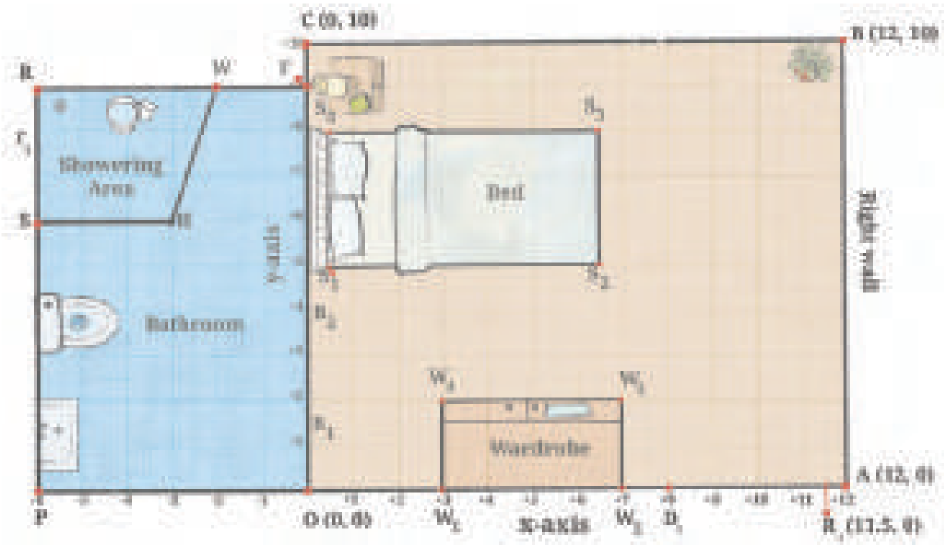


Fig. 1.5

1. Place Reiaan's rectangular study table with three of its feet at the points (8, 9), (11, 9) and (11, 7).
   (i) Where will the fourth foot of the table be?
   (ii) Is this a good spot for the table?
   (iii) What is the width of the table? The length? Can you make out the height of the table?
2. If the bathroom door has a hinge at B_1 and opens into the bedroom, will it hit the wardrobe? Are there any changes you would suggest if the door is made wider?
3. Look at Reiaan's bathroom.
   (i) What are the coordinates of the four corners O, F, R, and P of the bathroom?
   (ii) What is the shape of the showering area SHWR in Reiaan's bathroom? Write the coordinates of the four corners.
   (iii) Mark off a 3 ft × 2 ft space for the washbasin and a 2 ft × 3 ft space for the toilet. Write the coordinates of the corners of these spaces.
4. Other rooms in the house:
   (i) Reiaan's room door leads from the dining room which has the length 18 ft and width 15 ft. The length of the dining room extends from point P to point A. Sketch the dining room and mark the coordinates of its corners.
   (ii) Place a rectangular 5 ft × 3 ft dining table precisely in the centre of the dining room. Write down the coordinates of the feet of the table.

1.4 Distance Between Two Points in the 2-D Plane

We know how to find the distance between two points if they are on the axes or if they form a line segment parallel to the axes. For example, in Fig. 1.5 we can find the distances W_1W_2 and S_1S_2. What should we do if the segment joining the points is not parallel to either axis? We can use the Baudhāyana–Pythagoras theorem which you have studied in Grade 8. We can use this result to find the distance between any two points in the xy-plane.

Look at triangle ADM in Fig. 1.6.

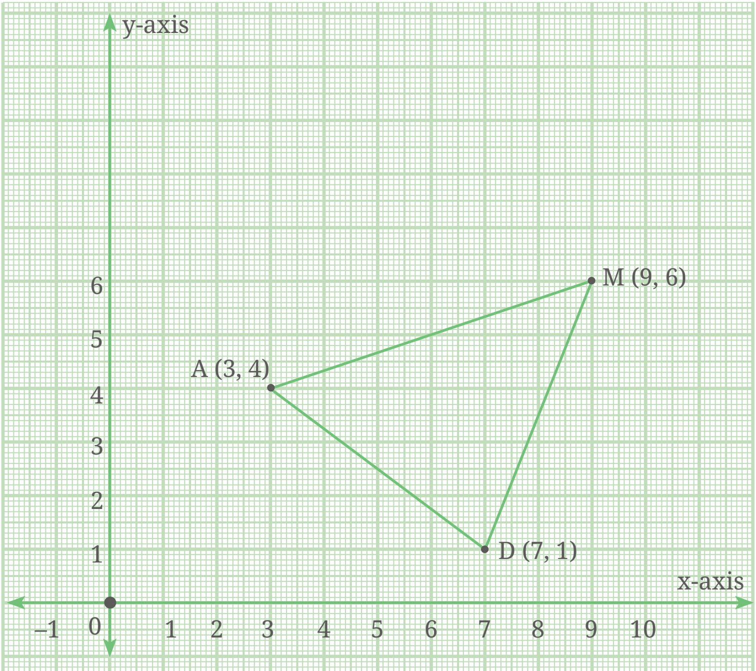


Fig. 1.6

Triangle ADM is an acute angled triangle in the first quadrant. How do we find the lengths of its sides AD, DM and MA?

Think and Reflect

1. In moving from A (3, 4) to D (7, 1), what distance has been covered along the x-axis? What about the distance along the y-axis?
2. Can these distances help you find the distance AD?

Fig. 1.7 gives us a clue. The distance moved along the x-axis is given by CD.

CD = x-coordinate of D – x-coordinate of A = 7 – 3 = 4.

The distance moved along the y-axis is given by AC.

AC = y-coordinate of A – y-coordinate of D = 4 – 1 = 3.

Using the Baudhāyana–Pythagoras Theorem, we get the distance

$$AD = \sqrt{4^2 + 3^2} = 5 \text{ units.}$$

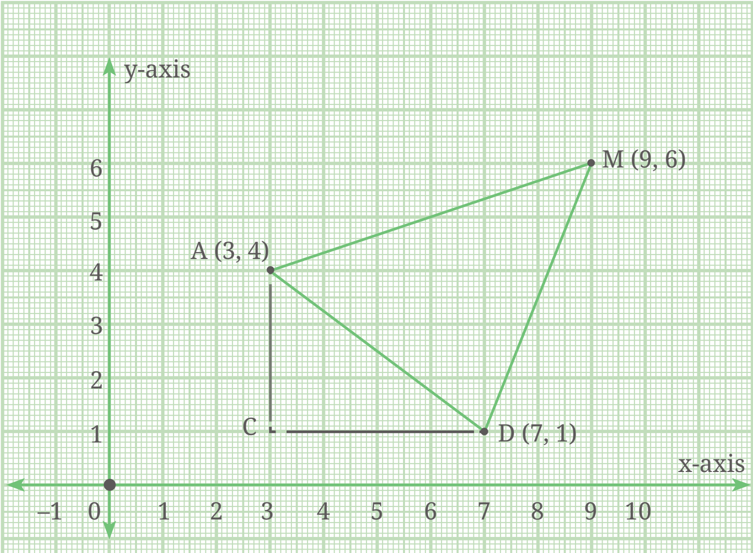


Fig. 1.7

We similarly find the distances DM and MA:

$$DM = \sqrt{2^2 + 5^2} = \sqrt{29} \text{ units.}$$

$$MA = \sqrt{6^2 + 2^2} = \sqrt{40} \text{ units.}$$

In general, the distance between the points (x_1, y_1) and (x_2, y_2) is given by

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

and is calculated as shown in Fig. 1.8.

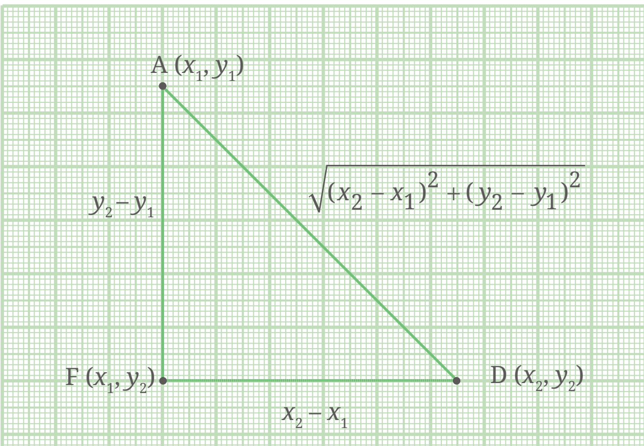


Fig. 1.8

It makes no difference whether $(x_2 - x_1)$ and $(y_2 - y_1)$ are positive or negative, as we are simply measuring the shifts along the two axes.

What if, x_1, x_2, y_1, y_2 take negative values? In Fig. 1.9, triangle AMD is reflected in the y-axis. What are the coordinates of the images of points A, M, and D?

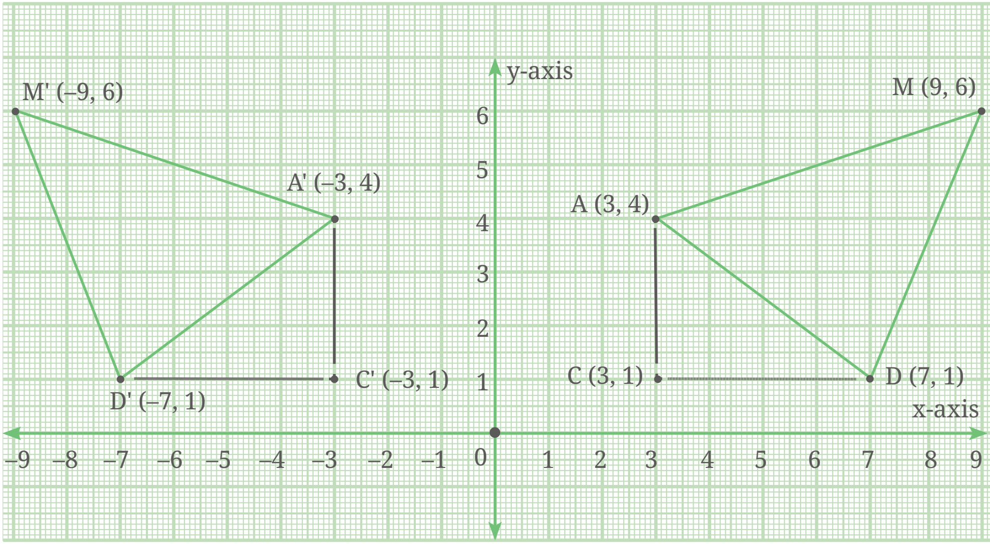


Fig. 1.9

$$C'D' = \text{x-coordinate of } A' - \text{x-coordinate of } D' = -3 - (-7) = 4.$$

$$A'C' = \text{y-coordinate of } A' - \text{y-coordinate of } D' = 4 - 1 = 3.$$

Using the Baudhāyana–Pythagoras Theorem, we get AD = $\sqrt{4^2 + 3^2} = 5$ units.

You can similarly calculate both D'M' and M'A':

$$D'M' = \sqrt{(-2)^2 + 5^2} = \sqrt{29} \text{ units}$$

$$M'A' = \sqrt{(-6)^2 + 2^2} = \sqrt{40} \text{ units.}$$

We see that reflection has preserved the lengths of the sides of the triangles.

Think and Reflect

1. What has remained the same and what has changed with this reflection?
2. Would these observations be the same if ΔADM is reflected in the x-axis (instead of the y-axis)?

End-of-Chapter Exercises

1. What are the x-coordinate and y-coordinate of the point of intersection of the two axes?
2. Point W has x-coordinate equal to –5. Can you predict the coordinates of point H which is on the line through W parallel to the y-axis? Which quadrants can H lie in?
3. Consider the points R (3, 0), A (0, –2), M (–5, –2) and P (–5, 2). If they are joined in the same order, predict:
   (i) Two sides of RAMP that are perpendicular to each other.
   (ii) One side of RAMP that is parallel to one of the axes.
   (iii) Two points that are mirror images of each other in one axis. Which axis will this be?

   Now plot the points and verify your predictions.
4. Plot point Z (5, –6) on the Cartesian plane. Construct a right-angled triangle IZN and find the lengths of the three sides.

   (**Comment:** Answers may differ from person to person.)
5. What would a system of coordinates be like if we did not have negative numbers? Would this system allow us to locate all the points on a 2-D plane?

*6. Are the points M (–3, –4), A (0, 0) and G (6, 8) on the same straight line? Suggest a method to check this without plotting and joining the points.

*7. Use your method (from Problem 6) to check if the points R (–5, –1), B (–2, –5) and C (4, –12) are on the same straight line. Now plot both sets of points and check your answers.

*8. Using the origin as one vertex, plot the vertices of:
   (i) A right-angled isosceles triangle.
   (ii) An isosceles triangle with one vertex in Quadrant III and the other in Quadrant IV.

*9. The following table shows the coordinates of points S, M and T. In each case, state whether M is the midpoint of segment ST. Justify your answer.

S	M	T	Is M the midpoint of ST? Yes or No	Reason for your answer
(–3, 0)	(0, 0)	(3, 0)		
(2, 3)	(3, 4)	(4, 5)		
(0, 0)	(0, 5)	(0, –10)		
(–8, 7)	(0, –2)	(6, –3)		

When M is the mid-point of ST, can you find any connection between the coordinates of M, S and T?

*10. Use the connection you found to find the coordinates of B given that M (–7, 1) is the midpoint of A (3, – 4) and B (x, y).

*11. Let P, Q be points of trisection of AB, with P closer to A, and Q closer to B. Using your knowledge of how to find the coordinates of the midpoint of a segment, how would you find the coordinates of P and Q? Do this for the case when the points are A (4, 7) and B (16, –2).

*12. (i) Given the points A (1, – 8), B (– 4, 7) and C (–7, – 4), show that they lie on a circle K whose center is the origin O (0, 0). What is the radius of circle K?

(ii) Given the points D (–5, 6) and E (0, 9), check whether D and E lie within the circle, on the circle, or outside the circle K.

*13. The midpoints of the sides of triangle ABC are the points D, E, and F. Given that the coordinates of D, E, and F are (5, 1), (6, 5), and (0, 3), respectively, find the coordinates of A, B and C.

14. A city has two main roads which cross each other at the centre of the city. These two roads are along the North–South (N–S) direction and East–West (E–W) direction. All the other streets of the city run parallel to these roads and are 200 m apart. There are 10 streets in each direction.

(i) Using 1 cm = 200 m, draw a model of the city in your notebook. Represent the roads/streets by single lines.

(ii) There are street intersections in the model. Each street intersection is formed by two streets — one running in the N–S direction and another in the E–W direction. Each street intersection is referred to in the following manner: If the second street running in the N–S direction and 5th street in the E–W direction meet at some crossing, then we call this street intersection (2, 5). Using this convention, find:

(a) how many street intersections can be referred to as (4, 3).

(b) how many street intersections can be referred to as (3, 4).

15. A computer graphics program displays images on a rectangular screen whose coordinate system has the origin at the bottom-left corner. The screen is 800 pixels wide and 600 pixels high. A circular icon of radius 80 pixels is drawn with its centre at the point A (100, 150). Another circular icon of radius 100 pixels is drawn with its centre at the point B (250, 230). Determine:

(i) whether any part of either circle lies outside the screen.

(ii) whether the two circles intersect each other.

16. Plot the points A (2, 1), B (–1, 2), C (–2, –1), and D (1, –2) in the coordinate plane. Is ABCD a square? Can you explain why? What is the area of this square?

Chapter Summary

- To locate the position of an object or a point in a plane, we require two perpendicular lines—One of them is horizontal, and the other is vertical.
- The plane is called the **cartesian plane**, the **coordinate plane** or the **xy-plane** and the lines are called the coordinate axes.
- The horizontal line is called the **x-axis** and the vertical line is called the **y-axis**.
- The coordinate axes divide the plane into four parts called **quadrants**.
- The point of intersection of the axes is called the **origin**.
- The distance of a point from the y-axis is its **x-coordinate** and the distance of the point from the x-axis is its **y-coordinate**. If the x-coordinate of a point is x, and the y-coordinate is y, then (x, y) are called the **coordinates** of the point.

- The coordinates of a point on the x-axis are of the form $(x, 0)$, and those of the points on the y-axis are of the form $(0, y)$.
- The coordinates of the origin are (0, 0).
- The coordinates of a point are of the form (+, +) in the first quadrant, (–, +) in the second quadrant, (–, –) in the third quadrant and (+, –) in the fourth quadrant.
- If $x = y$, then $(x, y) = (y, x)$. If $x \neq y$, then $(x, y) \neq (y, x)$.
- The distance between points (x_1, y) and (x_2, y) is the absolute value $|x_2 - x_1|$ of the difference between x_1 and x_2.
- The distance between points (x, y_1) and (x, y_2) is the absolute value $|y_2 - y_1|$ of the difference between y_1 and y_2.
- By the Baudhāyana–Pythagoras Theorem, the distance between points (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

2 Introduction to Linear Polynomials

2.1 Introduction

We have learnt about algebraic expressions in earlier grades. In this chapter, we will learn about the special types of algebraic expressions called **linear polynomials**. Let us first consider a few examples of algebraic expressions.

Example 1: Raju went to a shop where there were sealed boxes of different colours on sale. The shop owner told him that the red boxes have 4 pens each and the blue boxes have 5 pencils each. Now, if Raju bought x red boxes and y blue boxes, how can he quickly figure out the total quantity of pens and pencils? Also, if he got 3 extra pens free, how many pens and pencils did he get altogether?

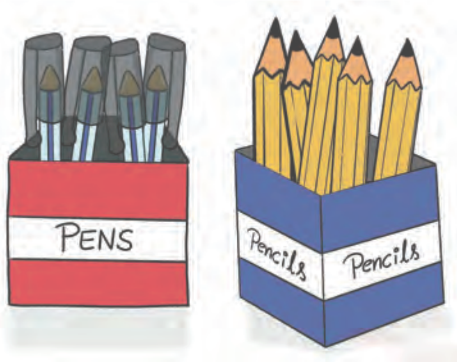


Fig. 2.1

Observe that x red boxes will have $4x$ pens and y blue boxes will have $5y$ pencils. Also, he got 3 extra pens free. Thus, the total number of pens and pencils is given by the algebraic expression $4x + 5y + 3$. In this example, $4x$, $5y$ and 3 are **terms** of the expression, x and y are letter-numbers, the numbers 4 and 5 are the **coefficients** of x and y, respectively, and 3 is a **constant**. From now onwards, we will use a widely used alternate word for letter-numbers: **variables**. Thus in the expression $4x + 5y + 3$, we say that the variables used are x and y.

Example 2: A rectangular garden of length l metres and width w metres has to be fenced and decorated. A wire fence is to be laid along the length costing ₹100 per metre and a wooden fence is to be built along the width costing ₹80 per metre. Special seeds have to be sown throughout the garden which will cost ₹50 per square metre.

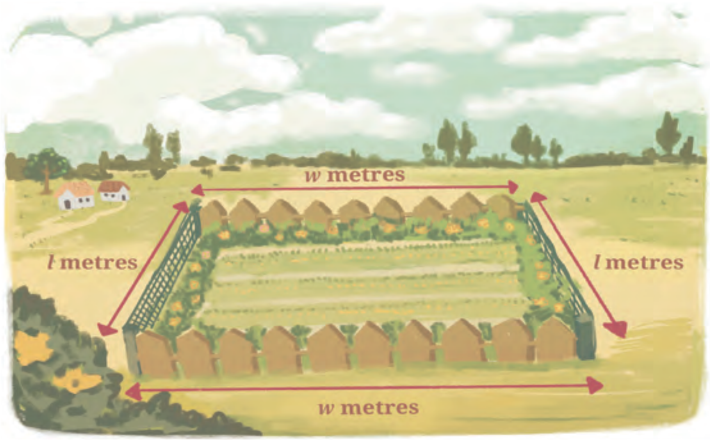


Fig. 2.2

What will be the total cost incurred?

Cost of wire fencing along the garden length = $2l \times 100$ = ₹$200l$

Cost of wooden fencing along the garden width = $2w \times 80$ = ₹$160w$

Cost of sowing seeds throughout the entire garden (depends on the area) = $50 \times l \times w$ = ₹$50lw$

Total cost = ₹ $(200l + 160w + 50lw)$.

Thus, $200l + 160w + 50lw$ is the algebraic expression for the total cost.

Think and Reflect

1. Can you identify the terms, variables and coefficients of this algebraic expression?
2. How is it different from the algebraic expression in Example 1?

Example 3: A wire of length 20 cm is bent in different ways to form rectangles. For example, we can have a rectangle with length 7 cm and width 3 cm. We can also have one of length 5.5 cm and width 4.5 cm. (Think of a few more ways of forming such rectangles.) Can you write an expression for the area of such rectangles?

If the length of the rectangle is x cm, then the width is $(10 - x)$ cm. The expression for the area of these rectangles is $x(10 - x)$ or $10x - x^2$.

Think and Reflect

1. Can you identify the terms, variables and coefficients of this algebraic expression?
2. Can you point out any similarity or difference between the algebraic expressions obtained in Examples 1 and 3?

Note that the algebraic expressions in Example 1 and Example 2 involve two variables, whereas the algebraic expression in Example 3 involves only one variable.

Expressions such as $4x$, $x^2 + 1$, $2y - 5$, $5y^3 + y^2 + 2y - 1$, $3z + 7$ are algebraic expressions that involve only one variable: x, y or z. In this chapter, we will restrict our discussion to algebraic expressions involving only one variable. You may have noticed that in an algebraic expression, the powers of a variable also appear. For example, in the expression $x^2 + 5x + 1$, the highest power of x is 2, whereas in the expression $5y^3 + y^2 - 8$, the highest power of the variable y is 3. Further, in the expression $5y^3 + y^2 + 2y - 1$, the coefficient of y^3 is 5, that of y^2 is 1, that of y is 2 and the constant term is –1. Such algebraic expressions involving one variable and its powers are called **one-variable polynomials**, **univariate polynomials**, or when the context is clear, simply **polynomials**. ('univariate' means 'having one variable'). The highest power of the variable in a polynomial is called its **degree**. For example:

(i) $5y^3 + y^2 + 2y - 1$ is a polynomial of degree 3. Such polynomials are called **cubic polynomials**.

(ii) $x^2 + 5x + 1$ is a polynomial of degree 2. Such polynomials are called **quadratic polynomials**.

(iii) $3z + 7$ is a polynomial of degree 1. Such polynomials are called **linear polynomials**.

(iv) The constant 8 is a polynomial of degree 0 as it can be written as $8x^0$ in which the power of the variable x is 0. Such polynomials are called **constant polynomials**.

Exercise Set 2.1

1. Find the degrees of the following polynomials:
   (i) $2x^2 - 5x + 3$
   (ii) $y^3 + 2y - 1$
   (iii) -9
   (iv) $4z - 3$
2. Write polynomials of degrees 1, 2 and 3.
3. What are the coefficients of x^2 and x^3 in the polynomial $x^4 - 3x^3 + 6x^2 - 2x + 7$?

4. What is the coefficient of z in the polynomial $4z^3 + 5z^2 - 11$?
5. What is the constant term of the polynomial $9x^3 + 5x^2 - 8x - 10$?

Recall that polynomials of degree 1 are called **linear polynomials**. In this chapter, we shall study linear polynomials.

2.2 Linear Polynomials

We begin with some examples involving linear polynomials.

Example 4: The perimeter of a square of side x is $4x$, which is a linear polynomial in the variable x.

Think and Reflect

Find the perimeter of squares with sides 1 cm, 1.5 cm, 2 cm, 2.5 cm and 3 cm. What will happen to the perimeters if the sides increase by 0.5 cm?

Example 5: A chess club charges a joining fee of ₹200 plus ₹50 for every match played. The following table shows the amount a player will have to pay as the number of matches varies.

Number of matches played	1	2	3	4	5	...	m
Amount paid (₹)	250	300	350	400	450	...	200 + 50m

Hence, if m is the number of matches played, the total cost will be ₹($200 + 50m$). Observe that $200 + 50m$ is a linear polynomial in the variable m. The amount paid increases by the constant value of ₹50 for every additional match played.

Think and Reflect

If a player paid ₹750, how many matches did he play?

The examples given above highlight a characteristic feature of linear polynomials — that the difference between the successive values at integers is constant. In Example 4, the perimeters increase by 2 cm each time the side of the square increases by 0.5 cm. Similarly in Example 5, the amount paid by a player increases by ₹50 for every additional match played. Such patterns are called **linear patterns**.

When we equate a linear polynomial in one variable to a constant, we get a **linear equation**. Let us consider the following example.

Example 6: The sum of two numbers is 64. One of the numbers is 10 more than the other. What are the two numbers?

Let the smaller number be x. Then the larger number must be $x + 10$. Since their sum is 64, we have the linear equation $x + (x + 10) = 64$. This implies that $2x + 10 = 64$. Note that $2x + 10$ is a linear polynomial. By equating it to 64 we get a linear equation $2x = 54$ or $x = 27$. The numbers are therefore, 27 and 37.

Polynomials can also be thought of as input-output processes. For instance, consider the linear polynomial $2x + 3$. For every x, there is a corresponding value of the polynomial $2x + 3$. For instance, if $x = 4$, we substitute 4 in the expression to get $2 \times 4 + 3 = 11$.

If $x = -6$, then we substitute -6 in the expression to get $2 \times -6 + 3 = -9$.

Fig. 2.3 shows this process as an input-output machine where the input is the value of x and the output is the value of $2x + 3$. This process can be referred to as a **function** where the expression $2x + 3$ is a function of the variable x. You will learn more about functions in later grades.

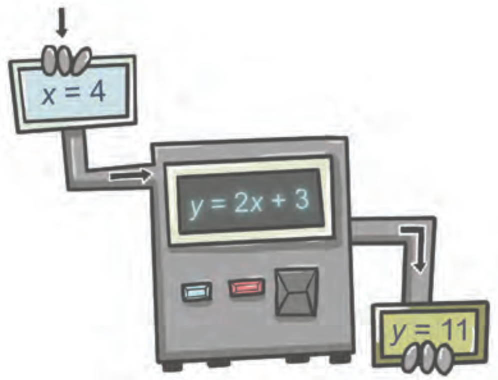


Fig. 2.3: A linear expression as an input-output process

Think and Reflect

We have learnt that to evaluate the value of an algebraic expression, we substitute a value of the variable in the given expression. Consider Example 3, where the wire is bent to form a rectangle. Here, the area of the rectangle, $10x - x^2$, is a function of x. Can you interpret this as an input-output process? What value does the expression take when $x = 6$ cm?

Note that $2x + 3$ is a linear function, whereas $10x - x^2$ is a quadratic function.

Exercise Set 2.2

1. Find the value of the linear polynomial $5x - 3$ if:

   (i) $x = 0$ (ii) $x = -1$ (iii) $x = 2$

2. Find the value of the quadratic polynomial $7s^2 - 4s + 6$ if:

   (i) $s = 0$ (ii) $s = -3$ (iii) $s = 4$

3. The present age of Salil's mother is three times Salil's present age. After 5 years, their ages will add up to 70 years. Find their present ages.
4. The difference between two positive integers is 63. The ratio of the two integers is 2:5. Find the two integers.
5. Ruby has 3 times as many two-rupee coins as she has five rupee-coins. If she has a total ₹88, how many coins does she have of each type?
6. A farmer cuts a 300 feet fence into two pieces of different sizes. The longer piece is four times as long as the shorter piece. How long are the two pieces?
7. If the length of a rectangle is three more than twice its width and its perimeter is 24 cm, what are the dimensions of the rectangle?

2.3 Exploring linear patterns

Observe the following growing pattern of square tiles.

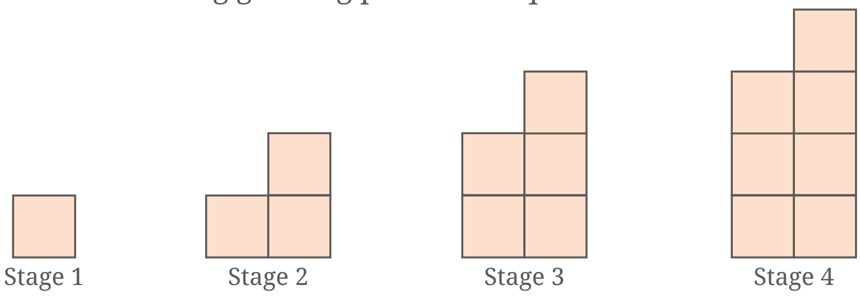


Fig. 2.4: A growing pattern of square tiles

Think and Reflect

Predict the number of squares in the next three stages of the pattern and write the sequence of numbers up to Stage 7 of the pattern.

Each stage is obtained by adding two more tiles to the previous stage. The table mentions the number of tiles for the first seven stages.

Stage	1	2	3	4	5	6	7
Number of the square tiles	1	3	5	7	9	11	13

To generalise this pattern, we observe that the number of squares at each stage is one less than twice the number of the term. For example, in Term 2, the number of squares is $2 \times 2 - 1 = 3$, in Term 5, the number of squares is $2 \times 5 - 1 = 9$ and so on.

This leads us to conclude that the number of squares at Stage n is given by $2n - 1$.

The polynomial $2n - 1$ has degree 1. Hence, it is an example of a linear polynomial. Also, the difference between consecutive terms in the sequence of the number of squares, that is, 1, 3, 5, 7, 9 ... is the constant value 2. Thus, with each stage, the number of squares increases by 2. The relationship between the number of the stage and the number of square tiles is a **linear relationship**.

Think and Reflect

Using the expression $2n - 1$, can you find out how many tiles will be there in the 15th stage and the 26th stage of the pattern? Also, which stage will contain 21 tiles and 47 tiles?

Example 7: Bela has ₹100 for pocket money. She spends ₹5 every day. After how many days will she be left with ₹40?

Day Number	0	1	2	3	4
Amount left (₹)	100	$100 - 1 \times 5$ $= 95$	$100 - 2 \times 5$ $= 90$	$100 - 3 \times 5$ $= 85$	$100 - 4 \times 5$ $= 80$

Observe that the amount left on the n^{th} day will be ₹(100 – 5n). Therefore, on the 12th day the amount left will be ₹(100 – 12 × 5) = ₹40.

Think and Reflect

What amount will be left on the 15th day? How many days will it take for the entire amount to be spent?

Example 8: An auto-rikshaw fare starts at ₹25 and remains the same for the initial 2 km. Then it increases by ₹15 per km. What will be the fare for a travel of 10 km?

For the initial 2 km, the fare is ₹25. For every kilometre (km) thereafter, the fare will increase by ₹15. Therefore, the total fare for a travel of 10 km will be ₹25 + 15 × 8 = ₹145.

Km travelled	1	2	3	4	5	6
Fare (₹)	25	25	25 + 1 × 15 = 40	25 + 2 × 15 = 55	25 + 3 × 15 = 70	25 + 4 × 15 = 85

Observe that the total fare for a travel of n km will be ₹25 + 15 × (n – 2) = 15n – 5, when $n \geq 2$. Here, the fare for a specific distance covered is a function of the distance n km.

Think and Reflect

For how many km will the fare be ₹130?

Note that in all the above examples, the n^{th} term is a linear expression in n. A **linear pattern** is a sequence of numbers where the difference between two consecutive terms is constant. We will learn more about linear patterns in the chapter on Sequences and Progressions.

Exercise Set 2.3

Solve the following:

1. A student has ₹500 in her savings bank account. She gets ₹150 every month as pocket money. How much money will she have at the end of every month from the second month onwards? Find a linear expression to represent the amount she will have in the n^{th} month.

2. A rally starts with 120 members. Each hour, 9 members drop out of the group. How many members will remain after 1, 2, 3, … hours? Find a linear expression to represent the number of members at the end of the n^{th} hour.
3. Suppose the length of a rectangle is 13 cm. Find the area if the breadth is (i) 12 cm, (ii) 10 cm, (iii) 8 cm. Find the linear pattern representing the area of the rectangle.
4. Suppose the length of a rectangular box is 7 cm and breadth is 11 cm. Find the volume if the height is (i) 5 cm, (ii) 9 cm, (iii) 13 cm. Find the linear pattern representing the volume of the rectangular box.
5. Sarita is reading a book of 500 pages. She reads 20 pages every day. How many pages will be left after 15 days? Express this as a linear pattern.

2.4. Linear growth and linear decay

Linear expressions help to model situations or phenomena where there is growth or decline. Consider the following examples:

Example 9: The cost of a journey is given by the linear function $C(d) = 100 + 60d$, where C indicates total cost in rupees and d the distance travelled in km. Let us make a table of values for d varying from 0 to 10 km and show how the cost increases for every km.

Distance travelled, *d* (km)	0	1	2	3	4	5
Cost, *C* (₹)	100	160	220	280	340	400

In this example, as the value of d increases by one km, the value of the cost function C, increases by a fixed amount of ₹60. This is an example of **linear growth**.

Think and Reflect

What is the cost for travelling 15 km? For how many kilometres will the cost of the journey be ₹700?

Example 10: The height of water in a cylindrical tank is 3 m at the start of summer. The height h m at the end of t months is given by the linear function $h(t) = 3 – 0.5t$.

Month, *t*	0	1	2	3	4
Height, *h* (*m*)	3	2.5	2	1.5	1

In the example above, as the value of t increases by a fixed number (one month), the value of the height h decreases by a fixed number (0.5). Therefore, this example represents **linear decay**.

Think and Reflect

What will be the height of the water at the end of 5 months?

Linear growth describes a linear pattern where a quantity increases by a constant amount over equal intervals. Similarly, linear decay describes a linear pattern where a quantity decreases by a constant amount over equal intervals.

Exercise Set 2.4

1. Suppose a plant has height 1.75 feet and it grows by 0.5 feet each month.
   (i) Find the height after 7 months.
   (ii) Make a table of values for t varying from 0 to 10 months and show how the height, h, increases every month.
   (iii) Find an expression that relates h and t, and explain why it represents linear growth.
2. A mobile phone is bought for ₹10,000. Its value decreases by ₹800 every year.
   (i) Find the value of the phone after 3 years.
   (ii) Make a table of values for t varying from 0 to 8 years and show how the value of the phone, v, depreciates with time.
   (iii) Find an expression that relates v and t, and explain why it represents linear decay.
3. The initial population of a village is 750. Every year, 50 people move from a nearby city to the village.
   (i) Find the population of the village after 6 years.

(ii) Make a table of values for t varying from 0 to 10 years and show how the population, P, increases every year.

(iii) Find an expression that relates P and t, and explain why it represents linear growth.

4. A telecom company charges ₹600 for a certain recharge scheme. This prepaid balance is reduced by ₹15 each day after the recharge.

(i) Write an equation that models the remaining balance $b(x)$ after using the scheme for x days. Explain why it represents linear decay.

(ii) After how many days will the balance run out?

(iii) Make a table of values for x varying from 1 to 10 days and show how the balance $b(x)$, reduces with time.

2.5 Linear Relationships

A linear relationship represents the relationship between two variables x and y, and can be expressed as $y = ax + b$.

Let us revisit the growing pattern of square tiles in Fig. 2.4. If x represents the number of the term and y represents the numbers of square tiles, then the linear relationship between x and y is expressed as $y = 2x - 1$.

Sometimes, we may be required to find the linear relationship between two quantities.

Example 11: A telecom company charges a fixed monthly fee and an additional cost per GB of the internet data used. A student observes that when she used 10 GB, her bill was ₹350. When she used 20 GB, her bill was ₹550. If the monthly bill y depends on the amount of data used, x (in GB), according to the relation $y = ax + b$, find the values of a and b.

Note that x = number of GB of the internet data used and y = total monthly cost in Rupees.

To find the linear relationship $y = ax + b$, we note that when $x = 10$, $y = 350$. Also, when $x = 20$, $y = 550$. We substitute these in $y = ax + b$ to arrive at the following equations.

$$350 = 10a + b \text{ and } 550 = 20a + b$$

We solve these as follows: Let $b = 350 - 10a$ (from the first equation). We substitute this in the second equation to obtain

$$550 = 20a + (350 - 10a)$$

Thus, $550 = 10a + 350$ or $10a = 200$. So, a = 20. This leads to $b = 350 - 10a = 350 - 200 = 150$.

We substitute the values of a and b in the equation $y = ax + b$ to obtain $y = 20x + 150$.

Thus, $y = 20x + 150$ represents the linear relationship between y, the bill amount in Rupees, and x, the number of GB of the internet data used.

Think and Reflect

Can you guess what the numbers 20 and 150 in the equation $y = 20x + 150$ represent?

Exercise Set 2.5

1. A learning platform charges a fixed monthly fee and an additional cost per digital learning module accessed. A student observes that when she accessed 10 modules, her bill was ₹400. When she accessed 14 modules, her bill was ₹500. If the monthly bill y depends on the number of modules accessed, x, according to the relation $y = ax + b$, find the values of a and b.
2. A gym charges a fixed monthly fee and an additional cost per hour for using the badminton court. A student using the gym observed that when she used the badminton court for 10 hours, her bill was ₹800. When she used it for 15 hours, her bill was ₹1100. If the monthly bill y depends on the hours of the use of the badminton court, x, according to the relation $y = ax + b$, find the values of a and b.
3. Consider the relationship between temperature measured in degrees Celsius (°C) and degrees Fahrenheit (°F), which is given by °C = a °F + b. Find a and b, given that ice melts at 0 degrees Celsius and 32 degrees Fahrenheit, and water boils at 100 degrees Celsius and 212 degrees Fahrenheit.
   (**Hint:** When °C = 0, °F = 32 and when °C = 100, °F = 212. Use this information to find a and b, and thus, the linear relationship between °C and °F.)

2.6 Visualising Linear Relationships

As we have seen in many examples in this chapter, a linear pattern or relationship can be expressed in the form of an equation $y = ax + b$. Now we shall learn to plot such an equation as a straight line.

To plot a linear equation, we need to identify any two points on the line. For example, to plot $y = 2x + 1$, we may identify two points as follows:

When $x = 0, y = 1$, so (0, 1) is a point on the line. We can call this point A. We say that the x-coordinate of A is 0 and the y-coordinate is 1. Also, when $x = 3, y = 7$. Thus, B (3, 7) is another point on the line. We plot these points on the coordinate plane, join them and extend the line in both directions as shown in Fig. 2.5.

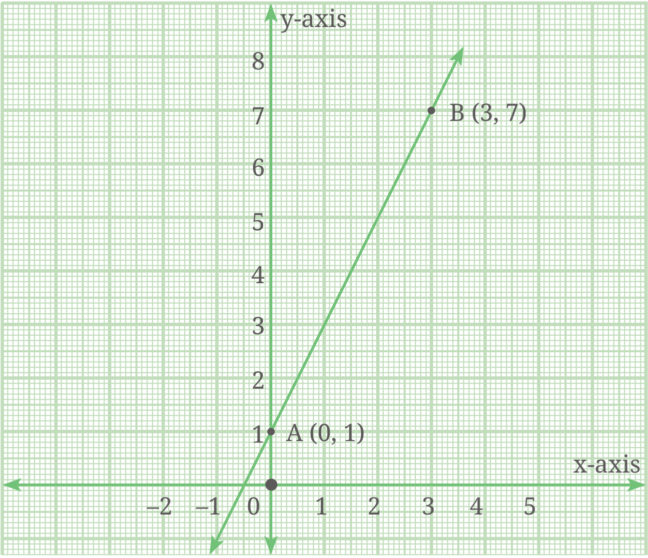


Fig. 2.5: The straight line $y = 2x + 1$

Think and Reflect

Identify other points on the line by completing the following table.

x	1	2	5	7	9	12	20
y	3			15			

Note that (1, 3) and (7, 15) are also the points on the line $y = 2x + 1$. Plot the points on the line. You may use a graph paper for this task. Observe that if a point lies on a line, its coordinates must satisfy the equation of the line. Thus, we can verify that (7, 15) lies on the line $y = 2x + 1$ by substituting $x = 7$ and $y = 15$ in the equation.

Example 12: Let us plot the points (–1, –3), (0, 0), (1, 3), (3, 9), (4, 12) in the coordinate plane on a graph paper as shown in Fig. 2.6. Join the points (–1, –3) and (4, 12) using a ruler. Doing so, observe that all five points lie on a straight line. Can you guess the equation of this line by looking at the relationship between the x and y coordinates of each point?

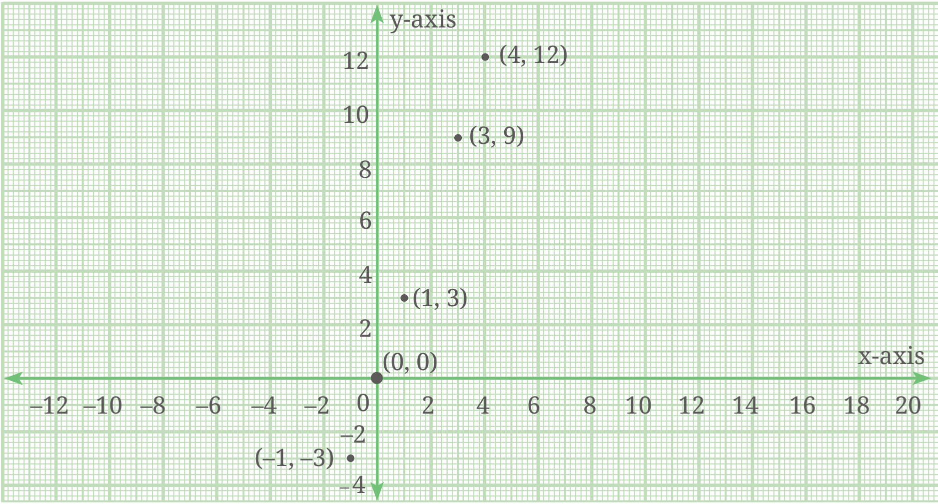


Fig. 2.6

For each point, the value of the y-coordinate is three times that of the x-coordinate. We can therefore say that $y = 3x$.

Example 13: Let us plot the points (–3, 6), (–2, 4), (0, 0), (1, –2), (2, –4), (3, –6) in the coordinate plane on a graph paper as shown in Fig. 2.7. Join the points (–3, 6) and (3, –6) using a ruler. Doing so, observe that all five points lie on a straight line. Can you guess the equation of this line by looking at the relationship between the *x* and *y* coordinates of each point?

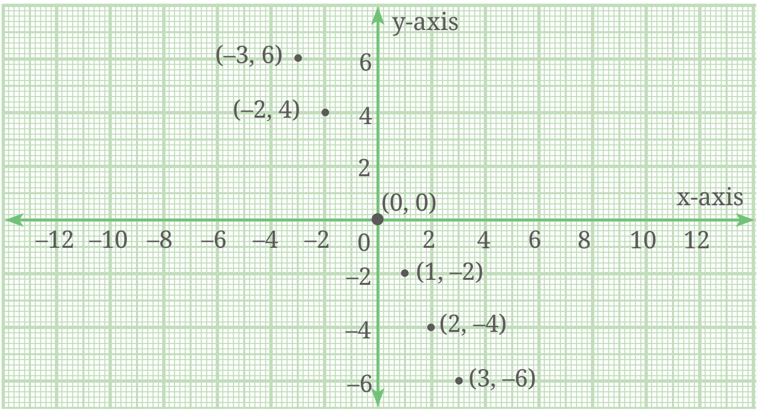


Fig. 2.7

Example 14: Draw the graphs of $y = \frac{1}{2}x$, $y = x$, $y = 2x$ by selecting suitable points on these lines.

(**Hint:** In order to graph $y = \frac{1}{2}x$, we could take the points (0, 0) and (4, 2). Can you verify that these lie on the line?)

Fig. 2.8 shows the graphs of these linear equations without any points labelled.

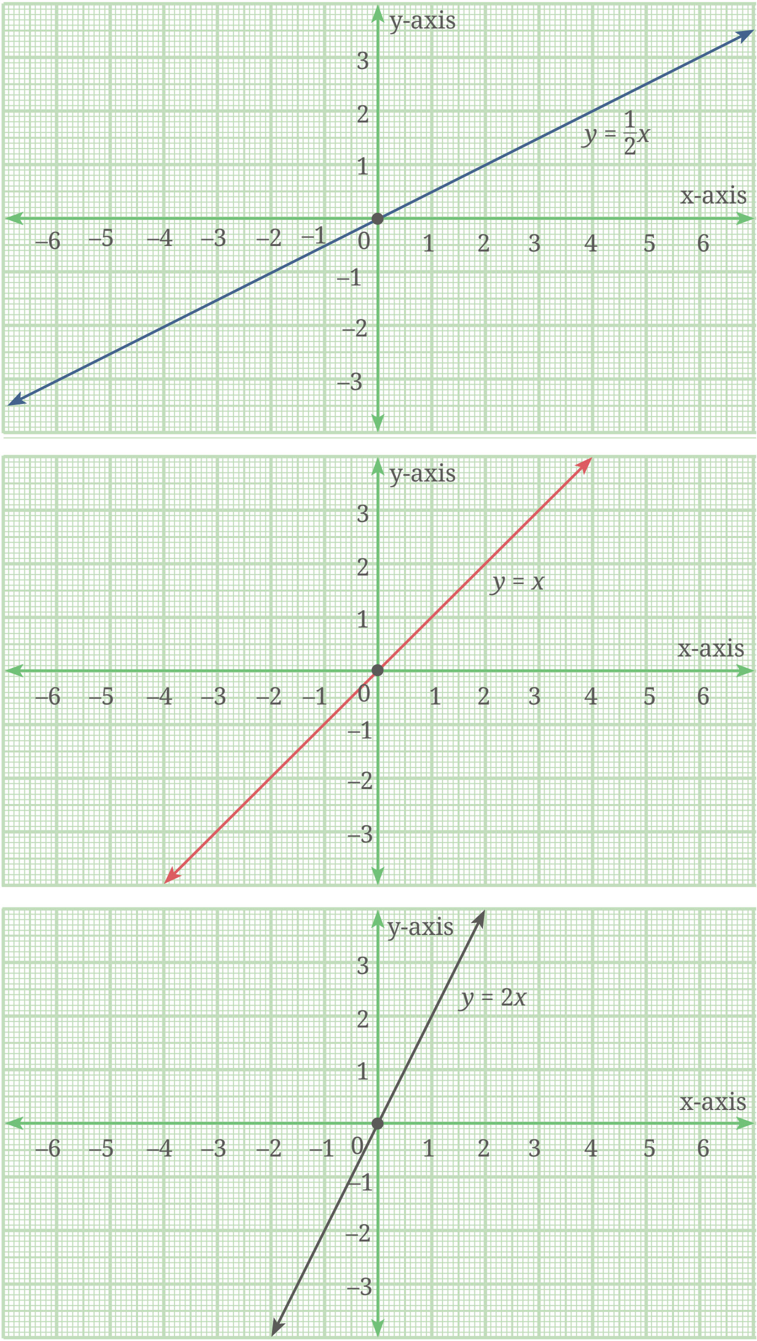


Fig. 2.8

Fig. 2.9 shows all the three graphs on the same axes. Does this help you to conclude anything about the linear equation y = ax, a > 0 as a varies? What happens when a > 1 and when a < 1?

(**Hint:** You may also plot the equations $y = 3x$ and $y = \frac{1}{3}x$ on the same axes.)

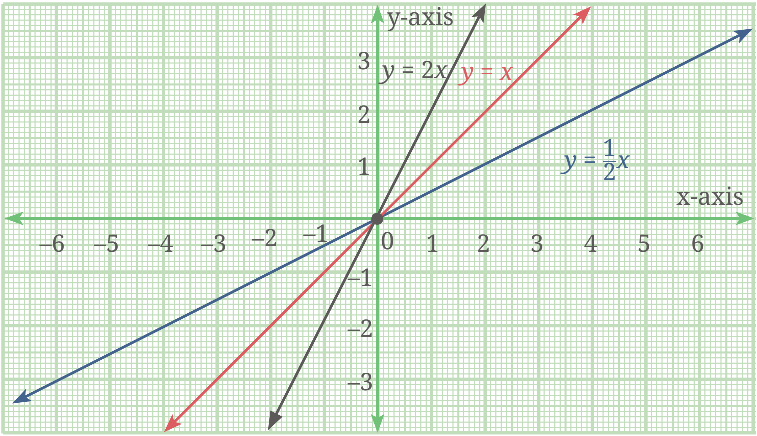


Fig. 2.9

First of all, we observe that the straight lines representing an equation of the form $y = ax$, always pass through the origin (0, 0). Further, when $a > 1$, the line is steeper than the line $y = x$, which is equally inclined to both axes. However, when $a < 1$, the line is less steep than the line $y = x$. In fact, a is referred to as the **slope** of the line $y = ax$. We will learn more about the concept of the **slope** of a line in the chapter on linear equations.

For now, we focus on another important observation. A linear growth is represented by a straight line with *positive* slope while a linear decay is represented by a straight line with negative slope. Note that the number of square tiles in the pattern in Fig. 2.4 leads to the sequence 1, 3, 5, 7, ... in which the difference of consecutive terms is the constant 2. The slope of the line representing the linear relationship between the term number and the number of square tiles is $y = 2x - 1$ is 2. Hence, the slope represents the constant difference between consecutive terms of the sequence.

Example 15: Now let us draw the graphs of $y = \frac{-1}{3}x$, $y = -x$, $y = -3x$ by selecting suitable points on these lines. Fig. 2.10 shows the graphs of these linear equations without any points labelled on them.

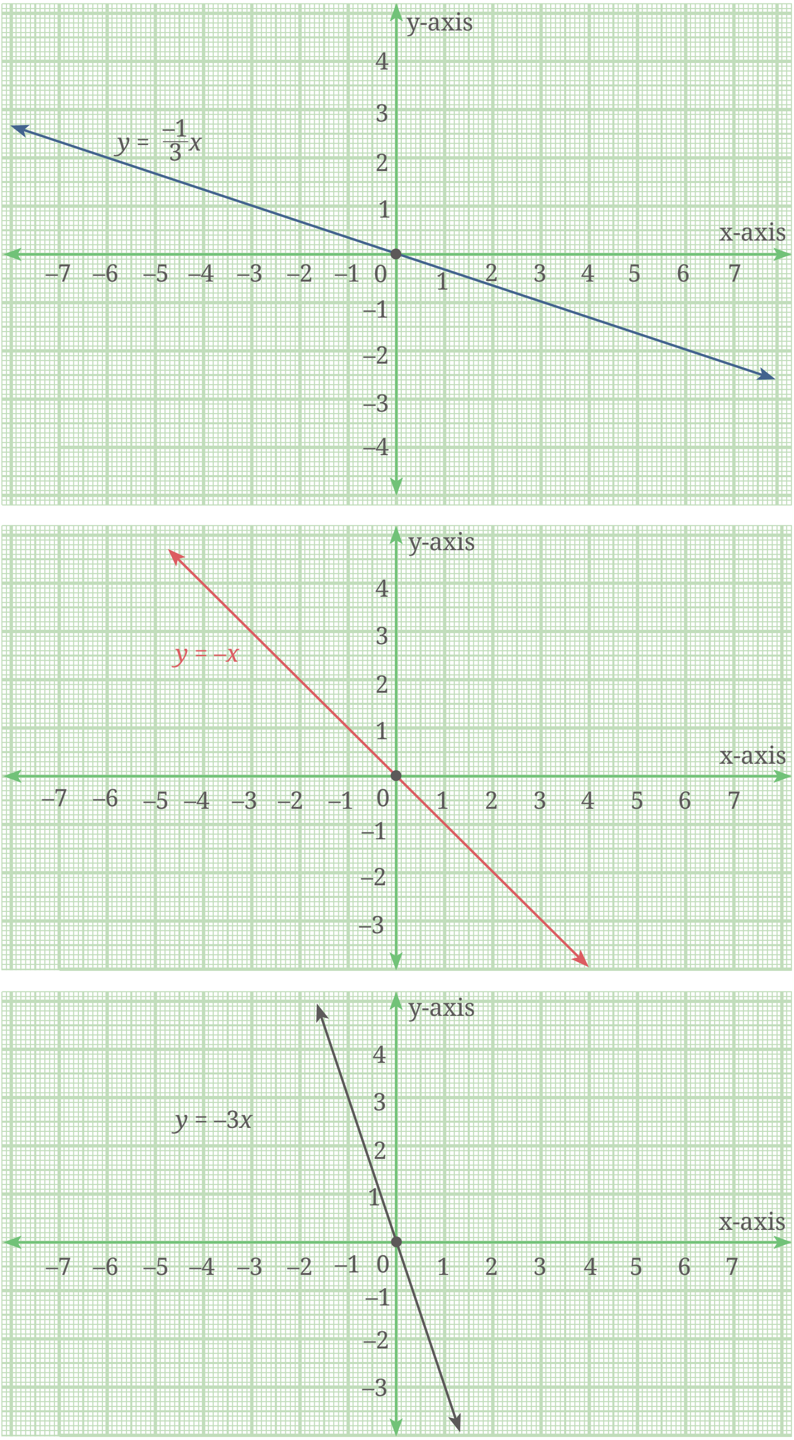


Fig. 2.10

Fig. 2.11 shows all the three graphs on the same axes. Does this help you to conclude anything about the linear equation $y = -ax$, $a > 0$, as a varies? What will happen when $a > 1$ and when $a < 1$?

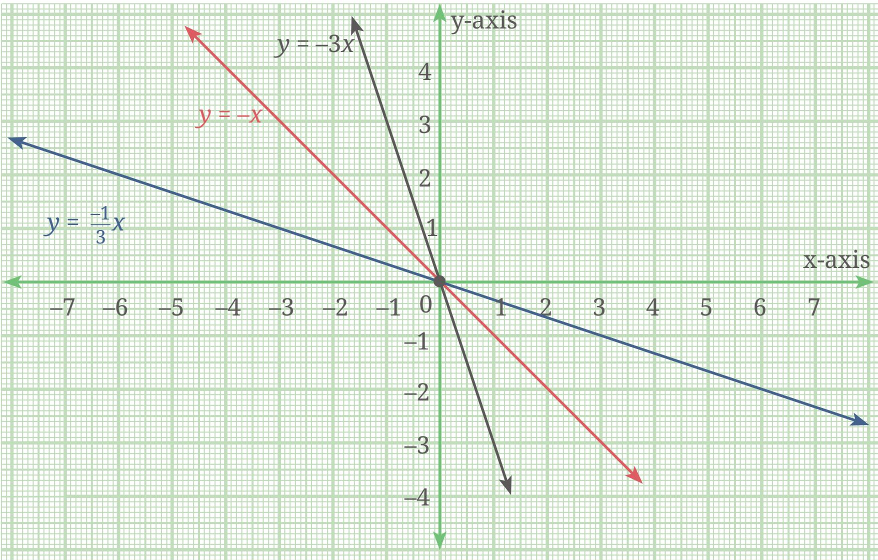


Fig. 2.11

Think and Reflect

Differentiate between the graphs of the equations $y = 3x + 1$, and $y = -3x + 1$.

Example 16: Let us now draw the graphs of $y = 2x - 1$, $y = 2x + 1$, $y = 2x + 5$, first individually (as shown in Fig. 2.12) and then on the same axes (as shown in Fig. 2.13).

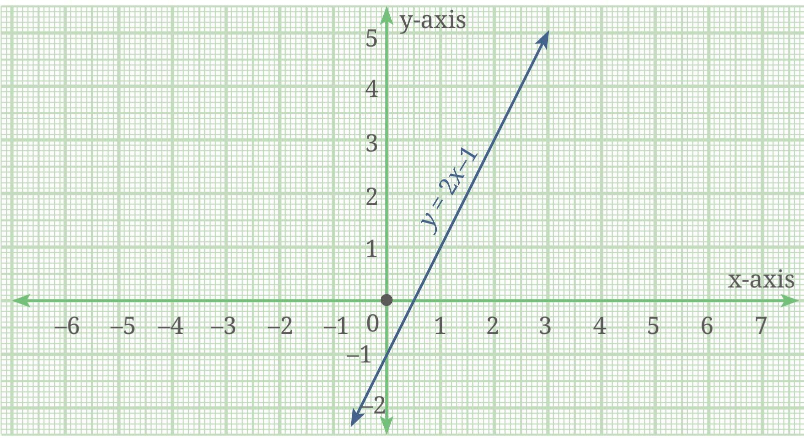


Fig. 2.12 A

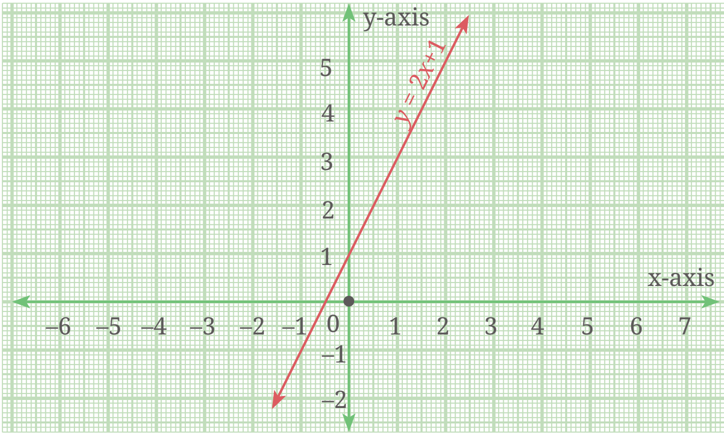


Fig. 2.12 B

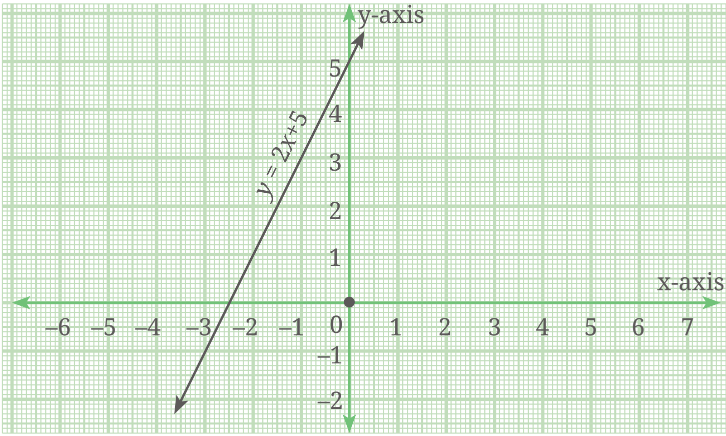


Fig. 2.12 C

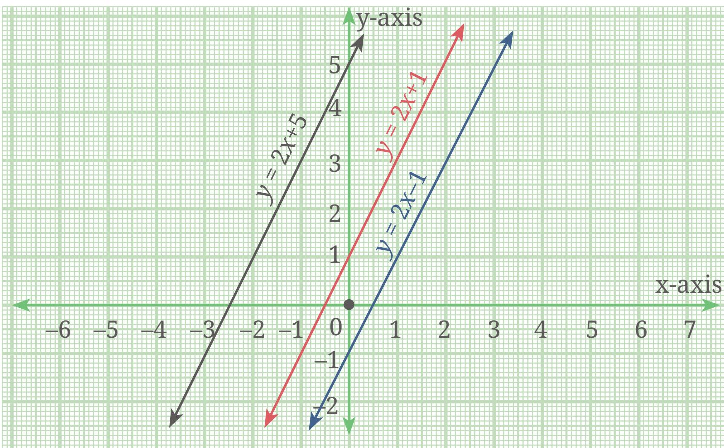


Fig. 2.13

Think and Reflect

Does this help you to conclude anything about the linear equation $y = ax + b$ when a is fixed but b varies?

(**Hint:** In these equations $a = 2$, and b takes the values –1, 1 and 5, respectively.)

Now let us draw the graphs of the equations $y = x + 3$, $y = 2x + 5$ and $y = 3x - 2$. See Fig. 2.14 and observe where these lines cut the y-axis.

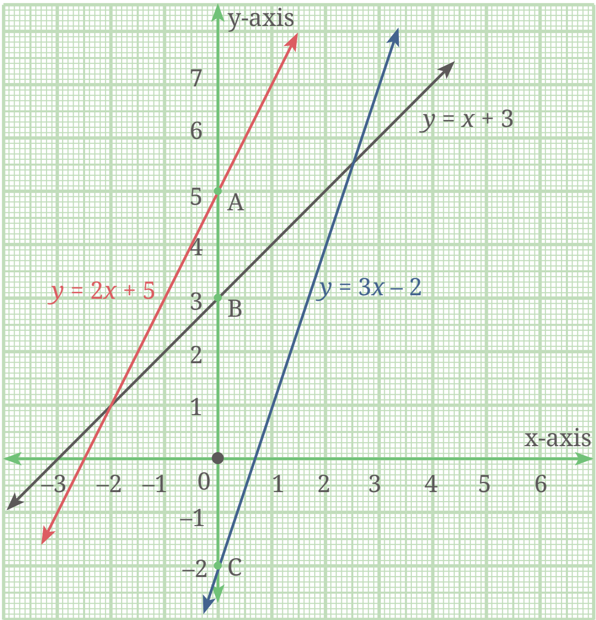


Fig. 2.14

We see that,

$y = 2x + 5$ cuts the y-axis at A (0, 5),

$y = x + 3$ cuts the y-axis at B (0, 3), and

$y = 3x - 2$ cuts the y-axis at C (0, –2).

We observe that any straight line written in the form $y = ax + b$ cuts the y-axis at the point (0, b). The length b is referred to as the y-intercept of the line. Thus, the y-intercept of the line $y = x + 3$ is 3. This means that the line cuts the y-axis at a distance of 3 units from the origin in the positive direction. Similarly, the y-intercept of the line $y = 3x - 2$ is – 2.

This means that the line cuts the y-axis at a distance of 2 units from the origin in the negative direction.

After observing the graphs in Figures 2.8 to 2.14, we may conclude the following:

(i) In the equation $y = ax + b$, a represents the slope of the line and b represents the y-intercept.

(ii) If we change the values of a, keeping b fixed, the slope of the line changes while the y-intercept remains fixed.

(iii) If we change the values of b keeping the value of a fixed, the lines shift but remain parallel to the original line. Thus, lines with equal slopes but different y-intercepts are parallel to each other.

Exercise Set 2.6

1. Draw the graphs of the following sets of lines. In each case, reflect on the role of 'a' and 'b'.

   (i) $y = 4x$, $y = 2x$, $y = x$

   (ii) $y = -6x$, $y = -3x$, $y = -x$

   (iii) $y = 5x$, $y = -5x$

   (iv) $y = 3x - 1$, $y = 3x$, $y = 3x + 1$

   (v) $y = -2x - 3$, $y = -2x$, $y = 2x + 3$

End-of-Chapter Exercises

1. Write a polynomial of degree 3 in the variable x, in which the coefficient of the x^2 term is –7.

2. Find the values of the following polynomials at the indicated values of the variables.

   (i) $5x^2 - 3x + 7$ if $x = 1$

   (ii) $4t^3 - t^2 + 6$ if $t = \mathrm{a}$

3. If we multiply a number by $\frac{5}{2}$ and add $\frac{2}{3}$ to the product, we get $\frac{-7}{12}$. Find the number.

4. A positive number is 5 times another number. If 21 is added to both the numbers, then one of the new numbers becomes twice the other new number. What are the numbers?

5. If you have ₹800 and you save ₹250 every month, find the amount you have after (i) 6 months (ii) 2 years. Express this as a linear pattern.

*6. The digits of a two-digit number differ by 3. If the digits are interchanged, and the resulting number is added to the original number, we get 143. Find both the numbers.

*7. Draw the graph of the following equations, and identify their slopes and y-intercepts. Also, find the coordinates of the points where these lines cut the y-axis.

   (i) $y = -3x + 4$

   (ii) $2y = 4x + 7$

   (iii) $5y = 6x - 10$

   (iv) $3y = 6x - 11$

   Are any of the lines parallel?

*8. If the temperature of a liquid can be measured in Kelvin units as x K and in Fahrenheit units as y °F, the relation between the two systems of measurement of temperature is given by the linear equation $y = \frac{9}{5}(x - 273) + 32$.

   (i) Find the temperature of the liquid in Fahrenheit if the temperature of the liquid is 313 K.

   (ii) If the temperature is 158 °F, then find the temperature in Kelvin.

*9. The work done by a body on the application of a constant force is the product of the constant force and the distance travelled by the body in the direction of the force. Express this in the form of a linear equation in two variables (work w and distance d), and draw its graph by taking the constant force as 3 units. What is the work done when the distance travelled is 2 units? Verify it by plotting it on the graph.

*10. The graph of a linear polynomial $p(x)$ passes through the points (1, 5) and (3, 11).

(i) Find the polynomial $p(x)$.

(ii) Find the coordinates where the graph of $p(x)$ cuts the axes.

(iii) Draw the graph of $p(x)$ and verify your answers.

*11. Let $p(x) = ax + b$ and $q(x) = cx + d$ be two linear polynomials such that:

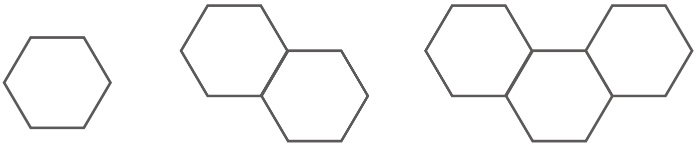
(i) $p(0) = 5$.

(ii) The polynomial $p(x) - q(x)$ cuts the x-axis at (3, 0).

(iii) The sum $p(x) + q(x)$ is equal to $6x + 4$ for all real x.

Find the polynomials $p(x)$ and $q(x)$.

*12. Look at the first three stages of a growing pattern of hexagons made using matchsticks. A new hexagon gets added at every stage which shares a side with the last hexagon of the previous stage.



Stage 1 Stage 2 Stage 3

(i) Draw the next two stages of the pattern. How many matchsticks will be required at these stages?

(ii) Complete the following table.

Stage Number	1	2	3	4	5	...	n
Number of matchsticks							

(iii) Find a rule to determine the number of matchsticks required for the n^{th} stage.

(iv) How many matchsticks will be required for the 15th stage of the pattern?

(v) Can 200 matchsticks form a stage in this pattern? Justify your answer.

*13. Let $p(x) = ax + b$ and $q(x) = cx + d$ be two linear polynomials such that:

(i) The graph of $p(x)$ passes through the points (2, 3) and (6, 11).

(ii) The graph of $q(x)$ passes through the point (4, –1).

(iii) The graph of $q(x)$ is parallel to the graph of $p(x)$.

Find the polynomials $p(x)$ and $q(x)$. Also, find the coordinates of the point where these lines meet the x-axis.

*14. What do all linear functions of the form $f(x) = ax + a$, $a > 0$, have in common?

Chapter Summary

- An **algebraic expression** combines numbers, variables, and operation symbols. For example, $2x^2 + 5xy - 3y^2$ is an algebraic expression in the variables x and y. $2x^2$, $5xy$ and $-3y^2$ are the **terms** of the algebraic expression, and the numbers 2, 5 and –3 are **coefficients** of the terms.
- **Univariate Polynomials** are algebraic expressions in one variable. Thus $x^2 + 5x + 3$ and $3y^3 - 4y^2 + 5$ are univariate polynomials in x and y, respectively. The highest power of the variable in a univariate polynomial is called its **degree**. Thus, $x^2 + 5x + 3$ is of degree 2 while $3y^3 - 4y^2 + 5$ is of degree 3.
- A polynomial of degree one is called a **linear polynomial**. Hence, $2x + 3$ and $5 - 4y$ are linear polynomials in the variables x and y, respectively.
- **Linear growth** refers to a pattern in which a quantity increases by a fixed amount over equal intervals. In contrast, **linear decay** describes a pattern in which a quantity decreases by a fixed amount over equal intervals.

- A **linear pattern** is a sequence of numbers where the difference between consecutive terms is constant.
- A **linear relationship** between two variables x and y is represented by a straight line $y = ax + b$. The slope of this line is a. The constant b is called the y-intercept which is the distance from the origin where the line cuts the y-axis. When $b = 0$, the equation of the line becomes $y = ax$ and the line passes through the origin.
- **Linear growth** is represented by a straight line with positive slope and **linear decay** is represented by a straight line with negative slope.
- Parallel lines are of the form $y = ax + b$, where the slope a is fixed while b, the y-intercept, varies.

3 The World of Numbers

3.1. The Dawn of Mathematics: The Human Need to Count

Long before humanity built cities, formulated laws, or studied the stars, there existed a fundamental, practical necessity: the need to keep count. Mathematics did not begin in a classroom with equations on a board; it began in the dirt, on the bark of trees, and on bones.

Imagine you are living thousands of years ago in a small agricultural settlement along the banks of the Saraswati river. You have a herd of cattle. Every morning, they go out into the dense forests to graze, and every evening, they return. How do you ensure that a calf has not wandered off? Without words for numbers, and without written symbols, early humans solved this through a concept called **one-to-one correspondence**.

For every cow that left the settlement, the herder might place one pebble in a clay pot. In the evening, for every cow that returned, one pebble was removed. If the pot was empty at the end of the day, the herd was safe. If pebbles remained, cows were missing. This simple act of matching one object to another was the birth of the **Natural Numbers** ($\mathbb{N} = \{1, 2, 3, 4, \dots\}$).

3.1.1 A History Written in Bone

While the decimal place-value system we use today was perfected in the Indian subcontinent, the earliest physical evidence of humanity recording natural numbers takes us deep into the heart of Africa. The first mathematicians did not use paper; they used tally marks carved into bone.

The **Lebombo Bone**, discovered in the Lebombo Mountains between South Africa and Swaziland, dates back approximately 35,000 years. It is a bone featuring 29 distinct, deliberately-carved uniformly-sized notches. Anthropologists and mathematicians believe this was not just random scratching, but a tool used as a lunar phase counter or a menstrual calendar, indicating that early humans were tracking time through natural numbers.

Even more fascinating is the **Ishango bone**, found near the headwaters of the Nile River in the Democratic Republic of Congo, dating to around 20,000 BCE. This bone contains three columns of asymmetrical notches. What makes the Ishango bone a mathematical marvel is the specific grouping of the tallies. One of the columns groups notches into 11, 13, 17, and 19 — the prime numbers between 10 and 20. Another column seems to demonstrate the concept of multiplication by 2 (doubling). These artefacts indicate that the abstract concept of a 'number' is indeed tens of thousands of years old.

Fig. 3.1: Representation of the prime number tally groupings found on the Ishango bone.

3.1.2 The Indian Context: Trade and Astronomy

As civilisations advanced, so did the need for larger numbers. In the ancient urban centers of the Indus Valley Civilisation, such as Lothal and Harappa, standardised weights and measures were crucial for trade. A merchant trading terracotta pottery, lapis lazuli, or cotton with Mesopotamia needed a robust system of accounting.

During Vedic times, Indian philosophers were fascinated by and deeply pondered large numbers. In the *Vedas*, which go back thousands of years, names were given to all powers of 10 up to 10^{12} (which was called *parārdha*). In the *Lalitavistara* in the 4th century BCE, Buddha describes names up to 10^{53}, which is called *tallakṣhaṇa*.

Expressing quantities in terms of powers of 10 was explicitly used in the *Ṛigveda*, thus setting the stage for the number system based on powers of 10 to be developed in India in the ensuing years, and which we now use around the world today. The development of the Indian numeral system in terms of place values and powers of 10 also helped pave the way for what is perhaps the most important mathematical invention in human history: the concept of zero.

Exercise Set 3.1

1. A merchant in the port city of Lothal is exchanging bags of spices for copper ingots. He receives 15 ingots for every 2 bags of spices. If he brings 12 bags of spices to the market, how many copper ingots will he leave with?
2. Look at the sequence of numbers on one column of the Ishango bone: 11, 13, 17, 19. What do these numbers have in common? List the next three numbers that fit this pattern.
3. We know that Natural Numbers are closed under addition (the sum of any two natural numbers is always a natural number). Are they closed under subtraction? Provide a couple of examples to justify your answer.

*4. Ancient Indians used the joints of their fingers to count, a practice still seen today. Each finger has 3 joints, and the thumb is used to count them. How many can you count on one hand? How does this relate to the ancient base-12 counting systems?

3.2 The Revolution of Śhūnya: When Nothing Became Something

For millennia, the number line started at 1. If you had five apples and gave all five away, you did not have a number to represent your state; you simply had a void, a lack of apples. Civilisations like the Babylonians and Mayans used placeholders — symbols to indicate an empty column in a number — but they did not treat 'nothing' as a number that you could add, subtract, and multiply.

It was in the work of Brahmagupta (628 CE) that the void was formally transformed into a number, which truly transformed mathematics. This monumental leap was in turn inspired by Indian philosophical traditions.

3.2.1 From Philosophy to Mathematics: The Concept of Śhūnyatā

In the *Upanishads* and in the vast Buddhist literature starting well before the 7th century BCE, the concept of **Śhūnyatā** (emptiness or nothingness) was a profound state that was the goal of yoga and meditation. The word *śhūnya* means zero, and the word **śhūnyatā** (zeroness) was used extensively in these works to describe the state that one is trying to reach during meditation—that of emptying one's mind of all *vṛttis* (fluctuations of the mind) to achieve the state of perfect stillness and tranquility. Patanjali in the Yoga Sutras around the 3rd century BCE also describes how *śhūnyatā* can help lead to control over one's mind, body, and senses.

Because Indian thinkers revered this state of 'emptiness', they possessed the conceptual framework necessary to welcome 'nothingness' as a concept in philosophy. This concept of zeroness then found its way into many fields, such as architecture, literature, linguistics, and eventually made its way into mathematics in the works of Āryabhaṭa and finally Brahmagupta.

Thus, the philosophical concept of emptiness crystallised into the mathematical zero.

3.2.2 The Bakhśhālī Manuscript and Brahmagupta's Rules

In the Hindu Number System that we use today, the physical transition from a blank space to a symbol can be seen in the **Bakhśhālī Manuscript**. Dated to the early centuries CE, it features a bold dot (*bindu*) used to represent zero.

However, a symbol is just a mark on a page until it has rules. The transformation of zero into a fully operational number occurred in the work of **Brahmagupta**. In his seminal work, the *Brāhmasphuṭasiddhānta* (628 CE), he explicitly defined zero as the result of subtracting a number from itself ($a - a = 0$).

Brahmagupta then laid down the fundamental laws of arithmetic with *śhūnya*:

Brahmagupta's Rules for Zero

- When zero is added to a number, the number remains unchanged: $a + 0 = a$.
- When zero is subtracted from a number, the number remains unchanged: $a - 0 = a$.
- When any number is multiplied by zero, the result is zero: $a \times 0 = 0$.

3.3 Integers: Expanding the Horizon

Brahmagupta did not stop at zero. He realised that if subtraction of a number from itself can result in zero (5 – 5 = 0), then what would happen if we subtracted a larger number from a smaller one (3 – 5 = ☐)?

To answer this, Brahmagupta grounded his mathematics in the reality of commerce and life.

He recognised two states:

- **Fortunes (*Dhana*):** Positive numbers, representing wealth or assets.
- **Debts (*Ṛiṇa*):** Negative numbers, representing debts.

By moving to the left of zero on the number line, Brahmagupta formally introduced **Negative Numbers** to the world. The combination of positive natural numbers, their negative counterparts, and zero creates the set of **Integers**, denoted by the symbol ℤ (from the German word *Zahlen*, meaning numbers).

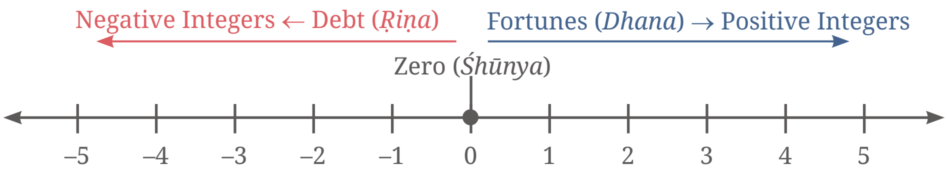


Fig. 3.2

3.3.1 The Arithmetic of Integers

Brahmagupta gave explicit rules for adding and multiplying these integers, which we still use exactly as he wrote them over 1,300 years ago:

1. **A fortune plus a fortune is a fortune:** 5 + 4 = 9.
2. **A debt plus a debt is a debt:** (–5) + (–4) = –9. (If you owe ₹5 and borrow ₹4 more, you owe ₹9.)
3. **A fortune minus zero is a fortune, a debt minus zero is a debt:** 7 – 0 = 7, and – 6 – 0 = – 6.
4. **The product of a debt and a fortune is a debt:** (–3) × 4 = –12. (If you take on 4 debts of ₹3, your total debt is ₹12.)
5. **The product of two debts is a fortune:** (–3) × (–4) = 12.

Think and Reflect

Why does a negative times a negative equal a positive? Think of it in terms of action and debt. If a negative number represents a debt, then multiplying by a negative number represents the removal of that debt.

(**Hint:** If someone takes away (–) four of your debts that are each worth ₹3 (that is, –3), you are effectively ₹12 richer! Therefore, (–3) × (–4) = +12.)

Exercise Set 3.2

1. The temperature in the high-altitude desert of Ladakh is recorded as 4 °C at noon. By midnight, it drops by 15 °C. What is the midnight temperature?
2. A spice trader takes a loan (debt) of ₹850. The next day, he makes a profit (fortune) of ₹1,200. The following week, he incurs a loss of ₹450. Write this sequence as an equation using integers and calculate his final financial standing.
3. Calculate the following using Brahmagupta's laws:
   (i) (–12) × 5 (ii) (–8) × (–7)
   (iii) 0 – (–14) (iv) (–20) ÷ 4
4. Explain, using a real-world example of debt, why subtracting a negative number is the same as adding a positive number (e.g., 10 – (–5) = 15).

3.4 Filling the Spaces: Fractions and Rational Numbers

As society grew more complex, measuring became just as important as counting. If a farmer divides a field of wheat among his three children, how much does each get? If a recipe calls for half a cup of ghee, how do we represent that mathematically?

Numbers that represent parts of a whole are called **fractions**. Just as every natural number has an additive inverse (3 has –3, 19 has –19, etc.), we can conceive of additive inverses for every positive fraction: $-\frac{3}{4}$ for $\frac{3}{4}$, $-\frac{19}{7}$ for $\frac{19}{7}$, etc. Let us refer to such additive inverses of positive fractions as negative fractions. Note that in a negative fraction, the '–' sign can also be placed with either the numerator or denominator, as follows: $-\frac{1}{5} = \frac{-1}{5} = \frac{1}{-5}$.

When we combine all integers and all fractions (both positive and negative), we get the set of **Rational Numbers**, denoted by $\mathbb{Q}$ (for quotient).

A **rational number** is defined as any number that can be expressed in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

Think and Reflect

Can you explain why we need $q \neq 0$ in the definition of a rational number?

We must make some important observations at this stage.

- All rational numbers can be written in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$. For example, 5 and –10 can be written as $\frac{5}{1}$ and $\frac{-10}{1}$, respectively. What this means is that the rational numbers also include the natural numbers, whole numbers and integers.
- Rational numbers do not have a unique representation in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$. For example, we have $-\frac{1}{3} = -\frac{2}{6} = -\frac{3}{9} = -\frac{10}{30} = -\frac{2026}{6078}$ and so on. These are equivalent rational numbers (or equivalent fractions). This fact allows us to freely divide any common factor between the numerator and the denominator of a given fraction. The resulting fraction is then equivalent to the original one. For example, $\frac{12}{30}$ is equivalent to $\frac{2}{5}$. Here, we have divided both the numerator and the denominator by 6.
- The common understanding is that when we say that $\frac{p}{q}$ is a rational number, or when we represent $\frac{p}{q}$ on the number line, we assume that $q \neq 0$ and that p and q have no common factors other than

1 (that is, p and q are **co-prime**). So, on the number line, among the infinitely many fractions equivalent to $\frac{1}{2}$ (i.e., the fractions $\frac{1}{2}$, $\frac{2}{4}, \frac{3}{9}, \frac{6}{12}, \ldots, \frac{1013}{2026}, \ldots$), we choose $\frac{1}{2}$ to represent all of them.

As you will remember from Grades 6 and 7, Brahmagupta also gave rules for addition, subtraction, multiplication, and division of fractions. He noted that these rules also apply to both positive and negative fractions which together constitute the rational numbers.

Here are the various laws:

1. **Equality:** Two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ are said to be equal if $ad = bc$.
2. **Addition and subtraction of two rational numbers**: We first express the two rational numbers as fractions $\frac{a}{b}$ and $\frac{c}{b}$ with the same denominator, b. Then we use the rules $\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}$ and $\frac{a}{b}-\frac{c}{b}=\frac{a-c}{b}$.
3. **Multiplication and division of two rational numbers**: We use the rules $\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$ provided that ($b \neq 0$, $d \neq 0$) and $\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}=\frac{ad}{bc}$ provided that ($b \neq 0$, $d \neq 0$, $c \neq 0$).
4. With the arithmetic laws defined as above, addition and multiplication are both commutative (i.e., $\frac{a}{b}+\frac{c}{d}=\frac{c}{d}+\frac{a}{b}$ and $\frac{a}{b}\times\frac{c}{d}=\frac{c}{d}\times\frac{a}{b}$, and they follow the law of distributivity: if p, q and r are rational numbers, then $p(q + r) = pq + pr$.

Rational numbers are closed under addition, subtraction, and multiplication; that is, if one adds two rational numbers, or subtracts two rational numbers, or multiplies two rational numbers, a rational number is again obtained. Rational numbers are also closed under division, provided that one does not divide by zero. That is, the quotient of two rational numbers is again a rational number, as long as one does not divide by zero.

Think and Reflect

1. While adding or subtracting two rational numbers having different denominators, how will you make the denominators equal?
2. Verify the distributive law for rational numbers.

Exercise Set 3.3

1. Prove that the following rational numbers are equal:

   (i) $\frac{2}{3}$ and $\frac{4}{6}$ (ii) $\frac{5}{4}$ and $\frac{10}{8}$

   (iii) $-\frac{3}{5}$ and $-\frac{6}{10}$ (iv) $\frac{9}{3}$ and 3

2. Find the sum:

   (i) $\frac{2}{5}+\frac{3}{10}$ (ii) $\frac{7}{12}+\frac{5}{8}$

   (iii) $-\frac{4}{7}+\frac{3}{14}$

3. Find the difference:

   (i) $\frac{5}{6}-\frac{1}{4}$ (ii) $\frac{11}{8}-\frac{3}{4}$

   (iii) $-\frac{7}{9}-\left(-\frac{2}{3}\right)$

4. Find the product:

   (i) $\frac{2}{3}\times\frac{3}{10}$ (ii) $\frac{7}{11}\times\frac{5}{8}$

   (iii) $-\frac{4}{7}\times\frac{5}{14}$

5. Find the quotient:

   (i) $\frac{2}{3}\div\frac{3}{10}$ (ii) $\frac{7}{11}\div\frac{5}{8}$

(iii) $-\frac{4}{7} \div \frac{5}{14}$

6. Show that: $\left(\frac{1}{2}+\frac{3}{4}\right)\times\frac{8}{3}=\frac{1}{2}\times\frac{8}{3}+\frac{3}{4}\times\frac{8}{3}$.

7. Simplify the following using the distributive property:
$\frac{7}{9}\left(\frac{6}{7}-\frac{3}{4}\right)$.

8. Find the rational number x such that: $\frac{5}{6}\left(x+\frac{3}{5}\right)=\frac{5}{6}x+\frac{1}{2}$.

3.4.1 Representation of Rational Numbers on the Number Line

We already know that integers can be represented on a number line. To do this, we first choose a point and mark it as 0, called the origin. Moving one unit to the right gives the point representing 1, two units to the right gives 2, and so on. Similarly, moving one unit to the left of the origin gives –1, two units to the left gives –2, and so on. Each integer lies at an equal distance from the next one. This is represented in Fig. 3.3.

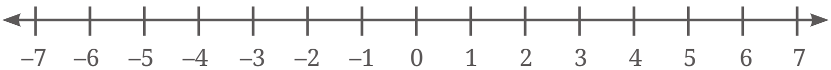


Fig. 3.3

Rational numbers can also be represented on the number line. Unlike integers, they may lie between two integers. For example, $\frac{1}{2}$ lies exactly halfway between 0 and 1, and $-\frac{3}{4}$ lies between –1 and 0 as shown in Fig. 3.4.

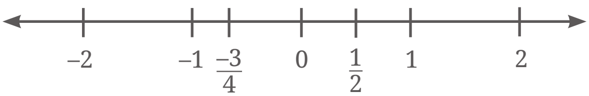


Fig. 3.4

To represent a rational number $\frac{p}{q}$, $q \neq 0$, on the number line, divide the unit interval (the distance between two consecutive integers) into q equal parts. Then move p parts from 0 to the right if the number is positive, and to the left if it is negative.

For example, to represent $\frac{3}{4}$, divide the interval between 0 and 1 into four equal parts and move three parts to the right from 0.

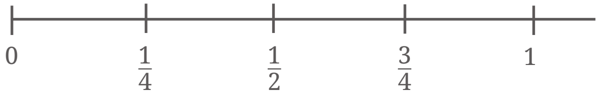


Fig. 3.5

Even fractions greater than 1 can be located on the number line in this way. For example, $\frac{9}{4} = 2\frac{1}{4}$ so it lies between 2 and 3. We divide the interval between 2 and 3 into four equal parts and move one part to the right of 2. See Fig. 3.6.

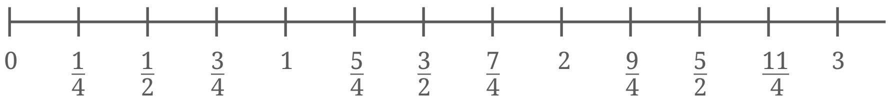


Fig. 3.6

Think and Reflect

Try and represent $\frac{8}{5}$ and $-\frac{7}{4}$ on a number line.

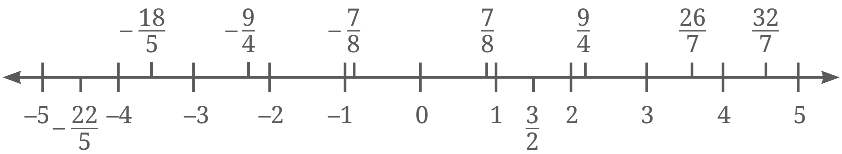


Fig. 3.7: Some integers and rational numbers on a number line

Absolute value of a rational number

The **absolute value** of a rational number x, written as $|x|$, represents its distance from 0 on the number line.

Example 1: $\left|\frac{5}{3}\right| = \frac{5}{3}$, $\left|-\frac{5}{3}\right|$ is also equal to $\frac{5}{3}$ and $|0| = 0$.

Thus, the absolute value of a positive number is the number itself. The absolute value of a negative number is its positive value. Therefore, the absolute value of any rational number is always non-negative, that is, $|x| \geq 0$.

For two rational numbers a and b, the distance between them on the number line is given by $|a - b|$. Fig. 3.8 represents the distance between the integers –4 and 3.

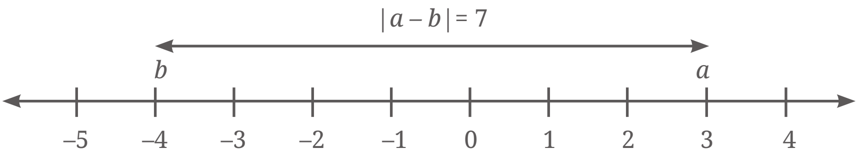


Fig. 3.8

3.4.2 The Density of Rational Numbers

One of the most magical properties of rational numbers is that they are dense. Between any two integers, say 1 and 2, there is a rational number $\frac{3}{2}$. Between 1 and $\frac{3}{2}$, there is another rational number $\frac{5}{4}$.

No matter how close two rational numbers are on the number line, you can always find another rational number between them by taking their average. For example, a rational number between 1 and $\frac{3}{2}$ can be found by taking their average: $\frac{1+\frac{3}{2}}{2} = \frac{5}{4}$. Try to explain why the average of two rational numbers a and b, which equals $\frac{(a+b)}{2}$, is always a rational number between a and b.

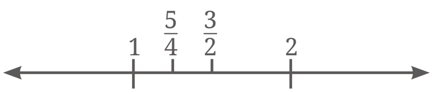


Fig. 3.9

This means that there are infinitely many rational numbers between any two points. It feels as though the rational numbers must completely fill the number line, leaving no gaps whatsoever. But do they?

Exercise Set 3.4

1. Represent the rational numbers $\frac{2}{3}$, $-\frac{5}{4}$ and $1\frac{1}{2}$ on a single number line.
2. Find three distinct rational numbers that lie strictly between $-\frac{1}{2}$ and $\frac{1}{4}$.

3. Simplify the expression: $\left(-\frac{1}{4}\right)+\left(\frac{5}{12}\right)$.
4. A tailor has $15\frac{3}{4}$ metres of fine silk. If making one kurta requires $2\frac{1}{4}$ metres of silk, exactly how many kurtas can he make?
5. Find three rational numbers between 3.1415 and 3.1416.

*6. Can you think of other way(s) to find a rational number between any two rational numbers?

3.5 Irrational Numbers

For centuries, mathematicians believed that every measurable length in the universe could be represented as a ratio of two integers. However, when Baudhāyana composed his *Śhulbasūtra* (a manual for constructing complex geometric fire altars) in around 800 BCE, he quickly encountered lengths that defied fractions. The ancient Greeks encountered the same crisis a few centuries later.

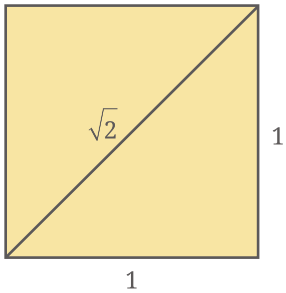


Fig. 3.10

Consider a square where each side is exactly 1 unit long. By the Baudhāyana–Pythagoras Theorem, the length of the diagonal d is given by $1^2 + 1^2 = d^2$, so $d^2 = 2$. Therefore, the length of the diagonal is $\sqrt{2}$.

Think and Reflect

Can $\sqrt{2}$ be written as a rational number $\frac{p}{q}$?

As you learned in Grade 8, the answer is a resounding NO! We will soon again see why! Numbers on the number line that cannot be expressed as a ratio of integers are called **Irrational Numbers**.

3.5.1 The Proof of Irrationality of $\sqrt{2}$

The first proof of the irrationality of $\sqrt{2}$ is due to a mathematician named Hippasus who was a member of the Pythagorean school (*c.* 400 BCE). To explain why $\sqrt{2}$ is irrational, he employed a brilliant technique called **Proof by Contradiction**. Namely, we assume the opposite of what we want to prove, and show that this assumption leads to a logical disaster.

Proof: We shall explain the proof in steps.

Step 1: Assumption: Assume $\sqrt{2}$ is a rational number. Therefore, it can be written as a fraction $\frac{p}{q}$, $q \neq 0$ in its simplest form. (This means p and q are integers that share no common factors other than 1, that is, they are co-prime.)

$$\sqrt{2} = \frac{p}{q}$$

Step 2: Square both sides of the equation:

$$2 = \frac{p^2}{q^2}$$

Step 3: Multiply both sides by q^2:

$$2q^2 = p^2$$

Step 4: Deduction for p:

Because p^2 is equal to 2 times some integer, p^2 must be an even number. If the square of a number is even, the number itself must be even. Therefore, p is an even integer. Let us say $p = 2k$ (where k is an integer).

Step 5: Substitute $p = 2k$ back into our equation from Step 3:

$$2q^2 = (2k)^2$$

$$2q^2 = 4k^2$$

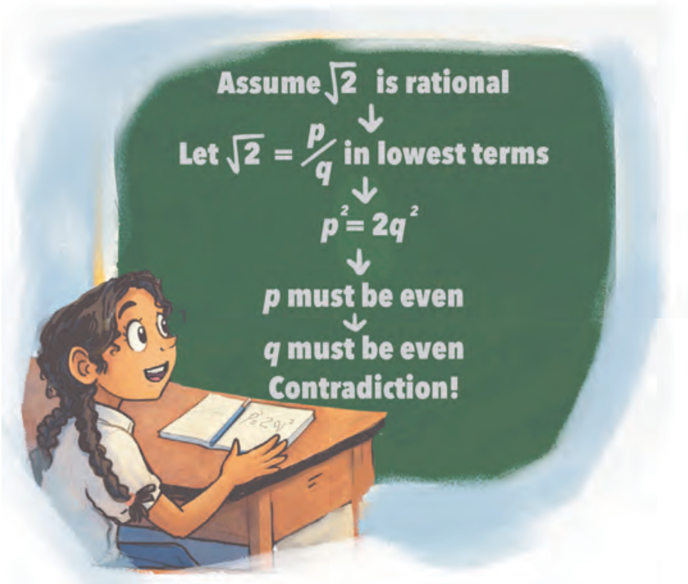
Step 6: Divide both sides by 2:

$$q^2 = 2k^2$$

Step 7: Deduction for q: Now we see that q^2 is equal to 2 times some integer k^2. This means q^2 is even, and therefore, q must also be an even integer.

Step 8: The Contradiction: We deduced that p is even and q is even. This means they both share a common factor of 2. However, in Step 1, we stated that $\frac{p}{q}$ was in simplest form, sharing no common factors!

Because our logical steps are flawless, our initial assumption must be wrong. Therefore, $\sqrt{2}$ cannot be expressed as a fraction. It is irrational.



Think and Reflect

Try to prove the irrationality of $\sqrt{3}$ using the approach of proof by contradiction. Will the same approach work for $\sqrt{5}$, $\sqrt{7}$, or $\sqrt{10}$?

3.5.2 Construction of Length $\sqrt{n}$

Think and Reflect

We have seen how to obtain a line whose length is a rational number. How do we obtain lines whose lengths are irrational?

For example, to construct a line segment of length $\sqrt{2}$ and mark its position on the number line, we may proceed as follows.

Step 1: On the number line (see Fig. 3.11), we measure OA = 1 unit and draw a perpendicular on OA through A.

Step 2: On this perpendicular line we mark the point B such that AB = 1 unit and join the origin to B. Clearly, OB = $\sqrt{2}$ units.

Step 3: With O as centre and OB as radius, with a compass we draw an arc which intersects the number line at P. Clearly, OP = $\sqrt{2}$ units. Hence, P represents the irrational number $\sqrt{2}$.

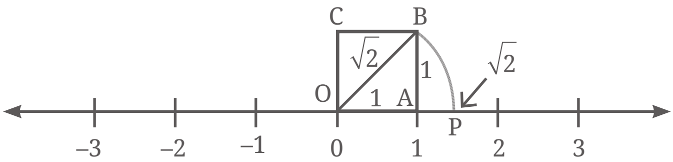


Fig. 3.11: Constructing irrational lengths and locating them on the number line

Think and Reflect

Try to extend this method for constructing line segments of lengths $\sqrt{3}$ and $\sqrt{5}$ using a ruler and a compass. Generalise this method to construct a line segment of any length of the form $\sqrt{n}$, where n is a positive integer.

3.5.3 The Story of Pi (π) and Madhava's Infinite Series

Another famous irrational number is π, the ratio of a circle's circumference to its diameter. For centuries, mathematicians sought better and better fractional approximations for π. Āryabhaṭa (499 CE) gave the highly accurate approximation $\frac{3927}{1250}$ = 3.1416, but stated that this was only an *asanna* (approximation), and indicated that an exact fraction could likely not be found.

However, because π is irrational (as was formally proven by Lambert in 1761), no single fraction (or finitely many fractions) can ever provide a perfect formula for π. An exact formula for π was first unlocked in the 14th century by **Mādhava of Sangamagrama**, who launched the Kerala School of Mathematics. Mādhava realised that to express an irrational number, you cannot use a single fraction; you must use an infinite sum. He discovered the profound infinite series:

$$\pi = 4 \times \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right).$$

We know what it means to add 2, 100, or even a lakh terms. What does it mean to add an infinite number of terms? It means finding the value that we get "closer" and "closer" to as we add more and more terms of the infinite sum, starting from the first term. You will learn more about infinite series in higher grades.

Thus, the number line is not only filled with rational numbers but also contains irrational numbers. We have seen how to mark some irrationals, and will simply conceive the rest of the irrationals to lie on the line.

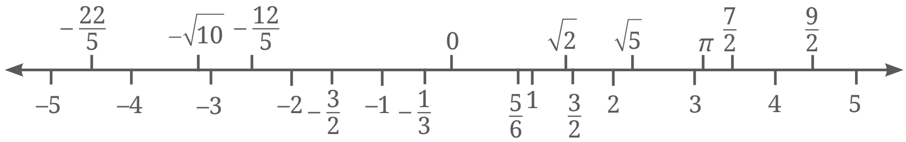


Fig. 3.12

3.6 Real Numbers: Decimals and Cyclic Patterns

When we unite the dense web of Rational Numbers with the unfillable gaps of Irrational Numbers, we create the unbroken, continuous line of the Real Numbers ($\mathbb{R}$).

The easiest way to distinguish a rational number from an irrational number is to look at its decimal expansion.

3.6.1 Rational Decimals: Terminating and Repeating

If you divide the numerator of a rational number by its denominator, exactly one of two things will happen:

1. **It terminates:** The division eventually leaves a remainder of 0. The decimal stops.

   Example 2: $\frac{3}{8} = 0.375$. (Can you tell for which rational numbers the decimal will be terminating?)

2. **It repeats:** The division never reaches a remainder of 0, but the sequence of digits begins to loop infinitely.

   Example 3: $\frac{5}{11} = 0.454545\ldots = 0.\overline{45}$.

Think and Reflect

Try to find the decimal expansions of $\frac{10}{3}$ and $\frac{11}{12}$. What do you observe about the repetition of the digits after the decimal point?

Why do some rational numbers have repeating decimal representations? Imagine calculating $\frac{1}{7}$ using long division. You are dividing by 7. What are the possible remainders at each step? They can only be 1, 2, 3, 4, 5, or 6 (why not 0?). Because there is a limited number of possible remainders eventually, a remainder must show up a second time. Once a remainder repeats, the entire division process loops, creating a repeating decimal!

Predicting the Type of Decimal Expansion

The interesting thing is that we can predict whether the decimal expansion of a rational number will terminate or go on forever without actually doing the long division process!

For this, we resort to finding the **prime factors** of the denominator.

Let $\frac{p}{q}$, $q \neq 0$, be a rational number in its lowest terms (that is, p and q have no common factor other than 1).

Consider the fraction $\frac{3}{20}$. The prime factorisation of 20 is $2^2 \times 5$.

Hence, $\frac{3}{20} = \frac{3}{2^2 \times 5} = \frac{3 \times 5}{2^2 \times 5 \times 5} = \frac{15}{100} = 0.15$.

We multiply the numerator and denominator by 5 to get 100 and arrive at 0.15 which is a terminating decimal.

Think and Reflect

The decimal expansion of $\frac{p}{q}$ will be terminating precisely when the prime factors of q are only 2, only 5 or both 2 and 5. Can you explain why?

This is because in such cases, we can make the denominator a power of 10 by multiplying both numerator and denominator by a suitable number.

Converting Rational Decimals into the Form $\frac{p}{q}$

Case 1: Terminating decimals

We have already seen in earlier grades how to convert terminating decimals into the form $\frac{p}{q}$.

Example 4: Convert 0.35 into the form $\frac{p}{q}$.

We know that $0.35 = \frac{35}{100} = \frac{7}{20}$.

Now, let us understand how to convert non-terminating repeating decimals into the form $\frac{p}{q}$.

Case 2: Pure repeating decimals

A pure repeating decimal is one in which a digit or a sequence of digits begins repeating immediately after the decimal point.

Example 5: Convert $0.\overline{6}$ into the form $\frac{p}{q}$.

Step 1: Let $x = 0.\overline{6}$

Step 2: Since one-digit repeats, multiply both sides by $10^1 = 10$, we get

$10x = 6.\overline{6}$

Step 3: Subtract the first equation from the second:

$10x - x = 6.\overline{6} - 0.\overline{6} = 6.$

Step 4: Solve for x: $9x = 6$. Thus, $x = \frac{6}{9} = \frac{2}{3}$.

Example 6: Convert $0.\overline{45}$ into the form $\frac{p}{q}$. Let $x = 0.\overline{45}$.

Since two digits repeat, multiply by $10^2 = 100$. We get $100x = 45.\overline{45}$. Subtracting the two equations, we get $99x = 45$.

So, $x = \frac{45}{99} = \frac{5}{11}$.

Case 3: General repeating decimals

A general repeating decimal has some non-repeating digits just after the decimal point, followed by a repeating block.

Example 7: Convert $0.1\overline{6}$ into the form $\frac{p}{q}$.

Step 1: Let $x = 0.1\overline{6}$.

Step 2: First, shift the decimal to place the repeating part immediately after the decimal point.

Since one digit is non-repeating, we multiply by $10^1 = 10$ and obtain $10x = 1.\overline{6}$

Step 3: Now, to move one full repeating cycle (1 digit repeating), we multiply by $10^1 = 10$ again and get $100x = 16.\overline{6}$.

Step 4: Subtracting the first shifted equation from the second, we get,

$100x - 10x = 16.\overline{6} - 1.\overline{6}$, or $90x = 15$. Thus, $x = \frac{15}{90} = \frac{1}{6}$.

Example 8: Convert $2.35\overline{7}$ into the form $\frac{p}{q}$.

Step 1: Let $x = 2.35\overline{7}$.

Step 2: Here, '35' is non-repeating (2 digits), '7' repeats (1 digit). Firstly, we multiply by $10^2 = 100$ to move the non-repeating digits before the decimal and we obtain $100x = 235.\overline{7}$.

Step 3: Now, we multiply by $10^1 = 10$ to move the full repeating cycle and get $1000x = 2357.\overline{7}$.

Step 4: Subtracting, we obtain $1000x - 100x = 2357.\overline{7} - 235.\overline{7}$.

Hence, $900x = 2122$ and $x = \frac{2122}{900} = \frac{1061}{450}$.

Example 9: Convert $2.45\overline{37}$ into the form $\frac{p}{q}$.

Step 1: Let $x = 2.45\overline{37}$.

Step 2: Here, '45' is non-repeating (2 digits), '37' repeats (2 digits). Firstly, we multiply by $10^2 = 100$ to move the non-repeating digits before the decimal and we obtain $100x = 245.\overline{37}$.

Step 3: Now, we multiply by $10^2 = 100$ to move the full repeating cycle and get $10000x = 24537.\overline{37}$.

Step 4: Subtracting, we obtain $10000x - 100x = 24537.\overline{37} - 245.\overline{37}$.

Hence, $9900x = 24292$ and $x = \frac{24292}{9900} = \frac{6073}{2475}$.

Summary Table for Conversion

Decimal type	Steps to follow
Pure repeating	• Let x = the decimal number. • Multiply by 10^n where n = the number of repeating digits, • Subtract from the original equation, and solve for x.

General repeating	• Let x = the decimal number. • Multiply by 10^m where m = the number of non-repeating digits, then multiply by 10^n where n = the number of repeating digits. • Subtract from the previous equation, and solve for x.

3.6.2 The Magic of Cyclic Numbers

When we calculate the decimal for $\frac{1}{7}$ we get 0.142857142857… or $0.\overline{142857}$. The repeating block, 142857, is one of mathematics' most fascinating gem — a **cyclic number**. Watch what happens when we multiply it by the digits 1 through 6:

$$142857 \times 1 = 142857$$
$$142857 \times 2 = 285714$$
$$142857 \times 3 = 428571$$
$$142857 \times 4 = 571428$$
$$142857 \times 5 = 714285$$
$$142857 \times 6 = 857142$$

The same digits simply shift in a cyclic circle! This beautiful internal structure is a hallmark of the rational number $\frac{1}{7}$.

3.6.3 Irrational Decimals: Chaos and Infinity

Irrational numbers, however, possess decimal expansions that never end and never repeat. There is no cyclic block, no pattern that loops forever. Examples include:

$$\sqrt{2} = 1.4142135623730950488\ldots$$
$$\pi = 3.14159265358979323 84\ldots$$

Exercise Set 3.5

1. Without performing long division, determine which of the following rational numbers will have terminating decimals and which will be repeating: $\frac{7}{20}$, $\frac{4}{15}$ and $\frac{13}{250}$. Then check your answers

by explicitly performing the long divisions and expressing these rational numbers as decimals.

2. Perform the long division for $\frac{1}{13}$. Identify the repeating block of digits. Does it show cyclic properties if you evaluate $\frac{2}{13}$? Now compute $\frac{3}{13}$, $\frac{4}{13}$, etc. What do you notice?

3. Classify the following numbers as rational or irrational:

   (i) $\sqrt{81}$

   (ii) $\sqrt{12}$

   (iii) 0.33333 ...

   (iv) 0.123451234512345 ...

   (v) 1.01001000100001 ... (Notice the pattern: Is it repeating a single block?)

   (vi) 23.560185612239874790120

   Find the explicit fractions in case they are rational.

4. The number $0.\overline{9}$ (which means 0.99999 ...) is a rational number. Using algebra (let $x = 0.\overline{9}$, multiply by 10, and subtract), explain why $0.\overline{9}$ is exactly equal to 1.

*5. We have seen that the repeating block of $\frac{1}{7}$ is a cyclic number. Try to find more numbers (n) whose reciprocals $\left(\frac{1}{n}\right)$ produce decimals with repeating blocks that are cyclic.

Non-uniqueness of decimal representations. Just as $1 = \frac{10}{10} = \frac{100}{100}$, rational numbers can have two decimal forms. Any terminating decimal has an alternative with repeating 9s: 1.000 ... = 0.999 ..., 2.47000 ... = 2.46999 ....

Is it not surprising that 0.999 ... = 1? Many would have guessed that it is slightly less than 1.

3.7 Conclusion: The Never-Ending Journey

We have travelled an immense distance. From the simple notches carved into the Ishango bone to track the passing days, to the profound philosophical void of *Śhūnya* in ancient India. We crossed the threshold of zero into the realm of debts and negative numbers with Brahmagupta. We found that the space between numbers is infinitely dense with fractions, yet fundamentally interspersed with the infinite-nonrepeating-decimal irrational numbers like $\sqrt{2}$ and π. By uniting the rational and the irrational, we built the continuous, unbroken **Real Number Line**. Every length, every temperature, every real physical measurement in the known universe has a home on this line.

The Evolution of Our World of Numbers

Our world of numbers has grown in stages over thousands of years, to meet the needs of humanity:

- **Natural Numbers (ℕ):** The basic counting numbers {1, 2, 3, …}. These are contained in the collection of integers.
- **Integers (ℤ):** These include also zero and the negative numbers {…, -2, -1, 0, 1, 2, …}. These in turn are contained in the collection of rational numbers.
- **Rational Numbers (ℚ):** These include all fractions $\frac{p}{q}$, where p, q are integers and $q \neq 0$. They also correspond to those numbers with terminating decimals like 0.135 and repeating decimals like $0.\overline{142857}$.
- **Irrational Numbers (𝕀):** These are separate from the above. They are numbers that cannot be written as fractions (e.g., $\sqrt{2}$, π, $\sqrt{10}$) and do not have terminating or repeating decimals.
- **Real Numbers (ℝ):** Together, the **rational** and **irrational** numbers make up the entire real number line.

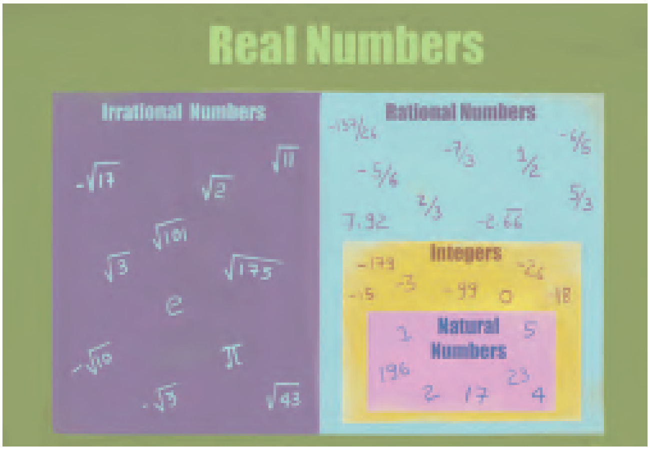


Fig. 3.13

Is the journey over? Are there more numbers waiting to be discovered beyond the Real Number line?

Think and Reflect

Consider this puzzle: What is the square root of –1? We know that 1 × 1 = 1. We also know that (–1) × (–1) = 1. There is no Real Number that, when multiplied by itself, results in a negative number. Thus, $\sqrt{-1}$ cannot exist on number line.

To solve this, mathematicians stepped completely off the line and invented a new dimension of numbers, denoted by the letter *i*, standing for **Imaginary Numbers**. While they sound like fiction, they are essential for modern electrical engineering, quantum mechanics, and the technology that powers your mobile phone. But that is a journey for another year. For now, master the Real Numbers. The universe is largely written in their language.

End-of-Chapter Exercises

1. Convert the following rational numbers in the form of a terminating decimal or non-terminating and repeating decimal, whichever the case may be, by the process of long division:

   (i) $\frac{3}{50}$ (ii) $\frac{2}{9}$

2. Prove that $\sqrt{5}$ is an irrational number.

3. Convert the following decimal numbers in the form of $\frac{p}{q}$.

   (i) 12.6 (ii) 0.0120 (iii) $3.0\overline{52}$

   (iv) $1.2\overline{35}$ (v) $0.\overline{23}$ (vi) $2.0\overline{5}$

   (vii) $2.12\overline{5}$ (viii) $3.12\overline{5}$ (ix) $2.\overline{1625}$

4. Locate the following rational numbers on the number line.

   (i) 0.532 (ii) $1.1\overline{5}$

5. Find 6 rational numbers between 3 and 4.

6. Find 5 rational numbers between $\frac{2}{5}$ and $\frac{3}{5}$.

7. Find 5 rational numbers between $\frac{1}{6}$ and $\frac{2}{5}$.

8. If $\frac{x}{3}+\frac{x}{5}=\frac{16}{15}$, find the rational number x.

9. Let a and b be two non-zero rational numbers such that $a+\frac{1}{b}=0$. Without assigning any numerical values, determine whether ab is positive or negative. Justify your answer.

10. A rational number has a terminating decimal expansion whose last non-zero digit occurs in the 4th decimal place. Show that such a number can be written in the form $\frac{p}{10^4}$, where p is an integer not divisible by 10. Is it necessary that the denominator of this rational number, when written in the lowest form, is divisible by 2^4 or 5^4? Give reasons.

11. Without performing division, determine whether the decimal expansion of $\frac{18}{125}$ is terminating or non-terminating. If it terminates, state the number of decimal places.

12. A rational number in its lowest form has denominator $2^3 \times 5$. How many decimal places will its decimal expansion have? Explain your answer.

*13. Let $a = \frac{7}{12}$ and $b = \frac{5}{6}$. Express both a and b in the form $\frac{k_1}{m}$ and $\frac{k_2}{m}$ where k_1, k_2 and m are integers and $k_2 - k_1 > 6$. Using the same denominator m, write exactly five distinct rational numbers lying between a and b keeping an integer numerator. Explain why the condition $k_2 - k_1 > n + 1$ is necessary to find n such rational numbers between the two rational numbers a and b using this method.

*14. Three rational numbers x, y, z satisfy $x + y + z = 0$ and $xy + yz + zx = 0$. Show that all the rational numbers x, y, z must be simultaneously zero.

*15. Show that the rational number $\frac{(a+b)}{2}$ lies between the rational numbers a and b.

16. Find the lengths of the hypotenuses of all the right triangles in Fig. 3.14 which is referred to as the square root spiral.

Fig 3. 14: Square root spiral

Chapter Summary

In this chapter you have learnt the following concepts:

- **Natural Numbers** (ℕ) are the counting numbers {1, 2, 3, … } that emerged at least tens of thousands of years ago due to humanity's need to count.
- The **Concept of Zero (Śhūnya)** was formalised in India by philosophers and then brought into mathematics formally by Brahmagupta (629 CE), who transformed the philosophical state of 'nothingness' (*Śhūnyatā*) into an actual number on which one could perform arithmetic operations. Brahmagupta also introduced the negative numbers, i.e., numbers less than zero.
- **Integers** (ℤ) extend the number line to the left of 1 to include zero as well as the negative numbers, which Brahmagupta historically categorised as 'debts' (*ṛiṇa*) in contrast with the positive number 'fortunes' (*dhana*).

- **Brahmagupta's Laws** provided the first rigorous framework for arithmetic with signed numbers, establishing rules such as 'the product of two debts is a fortune' ($- \times - = +$), as well as the rules for zero.
- **Rational Numbers** ($\mathbb{Q}$) are defined as any number that can be expressed as a ratio $\frac{p}{q}$ (where p and q are integers and $q \neq 0$). Brahmagupta also gave the formal rules for addition, subtraction, multiplication, and division of rational numbers.
- **Rational numbers** are **dense**, meaning a rational number always exists between any two other rational numbers.
- **Irrational Numbers** are values like $\sqrt{2}$ and π that cannot be written as fractions. Their existence proves that the number line contains gaps that rational ratios alone cannot fill. The first proof of the irrationality of a number was that of $\sqrt{2}$, by Hippasus (*c*. 400 BCE). It was a proof by contradiction. A proof that π is irrational (as it was suspected to be by Āryabhaṭa in 499 CE) was given by the Swiss mathematician Johann Lambert in 1761.
- **Real Numbers** ($\mathbb{R}$) represent the total union of all rational and irrational numbers, forming a perfectly continuous and unbroken line where every real physical measurement has a corresponding point.
- **Decimal expansions** serve as a mathematical signature for rational vs. irrational: rational numbers always result in terminating or repeating decimals, while irrational numbers produce non-repeating decimals that continue infinitely.
- **Cyclic Numbers**, such as the digits found in the repeating block of $\frac{1}{7}$ (i.e., 142857), reveal the elegant and symmetrical internal patterns hidden within the rational numbers.
- **Imaginary Numbers** are introduced as a final conceptual frontier to handle operations like $\sqrt{-1}$, which cannot be solved on the real number line and require a new dimension of mathematics.

4 Exploring Algebraic Identities

4.1 Introduction

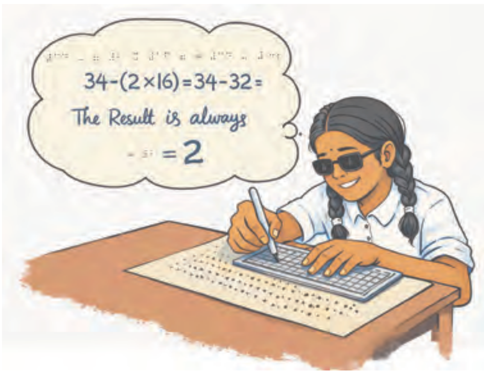
In earlier chapters, you learnt about linear polynomials and how they can be used to represent and solve real-life problems. You also studied linear equations and discovered how they describe relationships between quantities.

In this chapter, we will take the next step by exploring algebraic identities. These are special mathematical rules that not only make it easier to simplify complicated calculations but also help us work efficiently with algebraic expressions.

Let us begin by exploring a few simple patterns.

Example 1: Consider any three consecutive square numbers. For example, 1, 4, and 9. Add the smallest and the largest squares. Thus, 1 + 9 = 10. Then subtract twice the middle square from this sum. This leads to 10 – (2 × 4) = 10 – 8 = 2.

Now try the same process with another set of three consecutive square numbers. Say 9, 16, 25.



For example, consider the consecutive squares 25, 36, 49.

Applying the same rule we get (25 + 49) – (2 × 36) = 74 – 72 = 2.

Repeat this process with other sets of three consecutive square numbers. The result always seems to be 2!

The pattern may look surprising, but soon we will uncover the reason behind it using algebra.

Think and Reflect

Try and find other patterns like this one. For example, you could consider 4 consecutive squares and see if you can find a pattern.

4.2 Visualising Identities

In this section we will revisit some algebraic identities that we have studied in earlier grades and try to visualise them using geometrical models. In particular, we will use squares and rectangles to represent terms.

Consider two line segments of lengths a and b units, respectively, and make a longer line segment of length $(a + b)$ units as shown in Fig. 4.1.

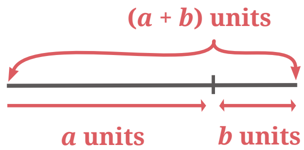


Fig. 4.1

We can now construct a square of side $(a + b)$ units and partition it into smaller squares and rectangles as shown in Fig. 4.2.

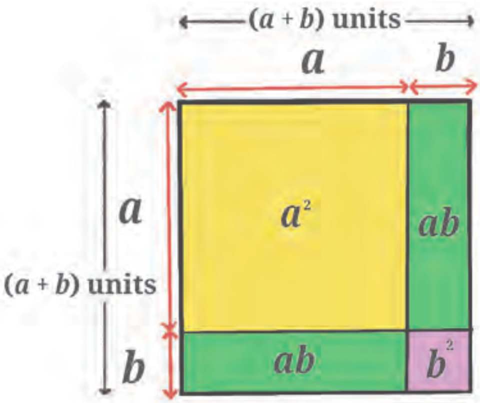


Fig. 4.2: Square of side (a + b) units

Observe that the area of the outer square is $(a + b)^2$. The area of the larger square inside the outer square is a^2 while the area of the smaller square is b^2. The areas of the two rectangles are ab each. Together they make the bigger square; hence we can conclude that

$$(a + b)^2 = a^2 + 2ab + b^2.$$

From Fig. 4.2 it is clear that $(a + b)^2 = a^2 + 2ab + b^2$ for all a and b when a and b are lengths of line segments.

Think of numbers a and b where a and b do not represent lengths of line segments. What if a and b are negative numbers? Let us check for some negative numbers and see if this equation still works.

Example 2: Let $a = –2$ and $b = –3$.

Then $(a + b) = –5$ and $(a + b)^2 = 25$.

Also $a^2 = 4$, $b^2 = 9$ and $2ab = 12$.

Thus $a^2 + 2ab + b^2 = 4 + 12 + 9 = 25$.

Hence, $a^2 + 2ab + b^2 = (a + b)^2$ again!

Now suppose a and b are rational numbers, say $a = -\frac{2}{3}$ and $b = \frac{3}{4}$

Then $(a + b) = \left(-\frac{2}{3} + \frac{3}{4}\right) = \frac{1}{12}$.

$$(a+b)^2 = \frac{1}{144}.$$

$$a^2 + 2ab + b^2 = \left(\frac{-2}{3}\right)^2 + 2\left(-\frac{2}{3}\right)\left(\frac{3}{4}\right) + \left(\frac{3}{4}\right)^2$$

$$= \frac{4}{9} - 1 + \frac{9}{16} = \frac{64 - 144 + 81}{144} = \frac{145 - 144}{144} = \frac{1}{144}.$$

So, $a^2 + 2ab + b^2 = (a + b)^2$ seems to be true for rational numbers too. But we are still not sure if it is true for all numbers. To verify this, let us investigate further using the distributive property of numbers:

$$(a + b)^2 = (a + b)(a + b) = a(a + b) + b(a + b)$$

$$= a^2 + ab + ba + b^2 = a^2 + 2ab + b^2.$$

Recall that in Grade 8 you were introduced to $(a + b)^2 = a^2 + 2ab + b^2$ as an *identity*.

What is the difference between an equation and an identity?

An **algebraic identity** is an equation that is true for all values of the variables occurring in it, while an equation need not be true for all values.

For example, $x^2 - 1 = 24$ is true for only $x = 5$ or -5. Hence, it is an equation.

But $(x + y)^2 = x^2 + 2xy + y^2$ is true for all values of x and y. Therefore, this equation is also an identity.

By now you must have observed that $(a + b)^2 \neq a^2 + b^2$. But can you find out which one of them will be greater?

For example, if $a = 10$ and $b = 2$, what are the values of $(a + b)^2$ and $a^2 + b^2$?

$(a + b)^2 = (10 + 2)^2 = 12^2 = 144$ and $a^2 + b^2 = 10^2 + 2^2 = 104$.

So in this case, $(a + b)^2 > a^2 + b^2$

But is it true for all numbers a and b?

Think and Reflect

1. What can you say about a and b if $(a + b)^2 < a^2 + b^2$?
2. What can you say about a and b if $(a + b)^2 > a^2 + b^2$?
3. When will $(a + b)^2$ be equal to $a^2 + b^2$?

Did you observe that $(a + b)^2$ and $a^2 + b^2$ are both positive? What term will decide which is larger? Use the expansion of $(a + b)^2$ to decide.

We can use the identity $(a + b)^2 = a^2 + 2ab + b^2$ to find the square of general binomial expressions.

Example 3: Let us try to expand $(5x + 2y)^2$. Here $a = 5x$ and $b = 2y$.

Then $(5x + 2y)^2 = (5x)^2 + 2(5x)(2y) + (2y)^2 = 25x^2 + 20xy + 4y^2$.

We can also use the identity to help us with numerical calculations.

Example 4: To calculate 43^2, we can write it as

$(40 + 3)^2 = 40^2 + 2 \times 40 \times 3 + 3^2 = 1600 + 240 + 9 = 1849$.

Exercise Set 4.1

1. Using the identity $(a + b)^2 = a^2 + 2ab + b^2$, expand the following:

(i) $(7x + 4y)^2$ (ii) $\left(\frac{7}{5}x + \frac{3}{2}y\right)^2$ (iii) $(2.5p + 1.5q)^2$

(iv) $\left(\frac{3}{4}s+8t\right)^2$ (v) $\left(x+\frac{1}{2y}\right)^2$ (vi) $\left(\frac{1}{x}+\frac{1}{y}\right)^2$

2. Using the same identity, find the values of the following:

(i) $(64)^2$ (ii) $(105)^2$ (iii) $(205)^2$

4.3 Factorisation of Algebraic Expressions Using Identities

The identity $(a + b)^2 = a^2 + 2ab + b^2$ can also be used to find factors of some algebraic expressions.

Example 5: Consider the algebraic expression $x^2 + 4x + 4$.

We observe that

$x^2 = (x)^2$, $4 = 2^2$ and $4x = 2(2)(x)$.

Hence, $x^2 + 4x + 4 = x^2 + 2(x)(2) + 2^2$ can be compared with $a^2 + 2(a)(b) + b^2$, where $a = x$ and $b = 2$.

We conclude that $x^2 + 4x + 4 = (x + 2)^2$. Therefore, we can say that $(x + 2)$ is a factor of $x^2 + 4x + 4$.

Example 6: Let us try to find factors of another algebraic expression:

$36x^2 + 12x + 1$.

Writing this in the form $a^2 + 2ab + b^2$ we get

$36x^2 + 12x + 1 = (6x)^2 + 2(6x)(1) + 1^2$,

where $a = 6x$ and $b = 1$. Therefore, $36x^2 + 12x + 1 = (6x + 1)^2$.

Thus $(6x + 1)$ is a factor of $36x^2 + 12x + 1$.

Example 7: Let us try to factor $50p^2 + 60pq + 18q^2$. What will a and b be in this case?

Try to think of a term whose square is $50p^2$. It is $\sqrt{50}p$. But if we want to avoid using the square root symbol we may proceed as follows:

We observe that 2 is a common factor of the terms

$50p^2$, $60pq$, $18q^2$.

Thus $50p^2 + 60pq + 18q^2 = 2(25p^2 + 30pq + 9q^2)$.

Now let us focus on the expression $25p^2 + 30pq + 9q^2$.

Think of a term whose square is $25p^2$. What about $9q^2$?

$50p^2 + 60pq + 18q^2$

$$= 2\left[(5p)^2 + 2(5p)(3q) + (3q)^2\right]$$

$$= 2\,(5p + 3q)^2.$$

Here we have used the identity $(a + b)^2 = a^2 + 2ab + b^2$ to factor the expression $25p^2 + 30pq + 9q^2$, after taking 2 as a common factor.

In all the examples described so far, we have used the identity $(a + b)^2 = a^2 + 2ab + b^2$ to find factors of different algebraic expressions.

Think and Reflect

What if we replace b by $-b$ in $(a + b)^2 = a^2 + 2ab + b^2$?

We get $(a - b)^2 = a^2 - 2ab + b^2$, which is also an identity and can be used in ways similar to $(a + b)^2 = a^2 + 2ab + b^2$.

Let us revisit the pattern we observed in Example 1. Any three consecutive numbers can be taken as $(n - 1)$, n and $(n + 1)$. Their respective squares are of the form $(n - 1)^2$, n^2 and $(n + 1)^2$.

The sum of the smallest and largest squares is

$$(n - 1)^2 + (n + 1)^2 = n^2 - 2n + 1 + n^2 + 2n + 1 = 2n^2 + 2.$$

Subtracting $2n^2$ from this leads to 2. Hence you always get 2! This is in fact a proof of the fact that if we add the smallest and largest of any three consecutive square numbers and subtract two times the middle square number, we will always arrive at 2.

Just as the identity $(a + b)^2 = a^2 + 2ab + b^2$ can be used to find the squares of numbers, we can also use $(a - b)^2 = a^2 - 2ab + b^2$ in a similar manner.

Example 8: Suppose we have to calculate 29^2. We can express this as $(30 - 1)^2 = 30^2 - 2 \times 30 \times 1 + 1^2 = 900 - 60 + 1 = 841$.

To visualise the identity $(a - b)^2 = a^2 - 2ab + b^2$, let us draw a square of side a units and split a in two parts, one of length $(a - b)$ units and another of length b units. Then the figure will look like this.

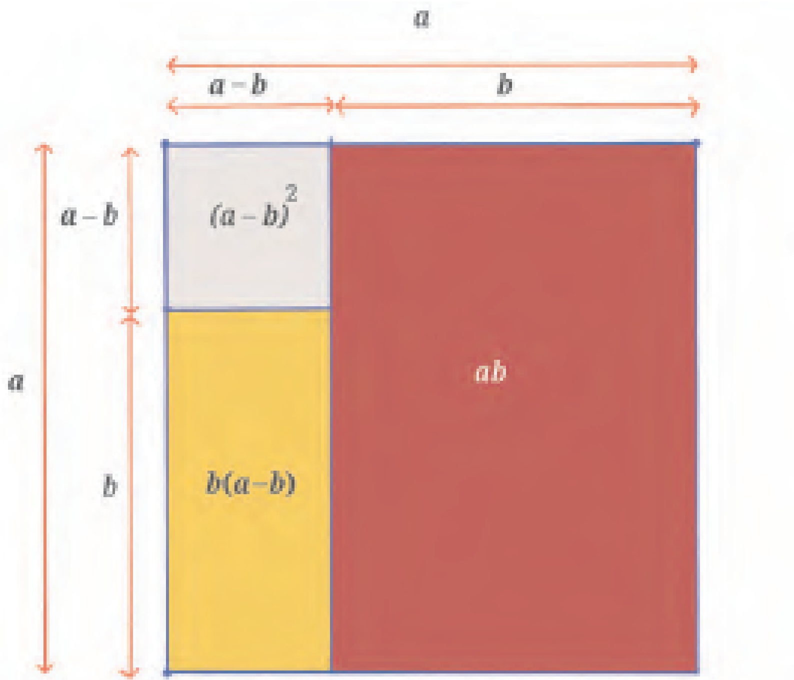


Fig. 4.3: A square of side a units

The area of the big square is a^2 square units. The small square has an area of $(a – b)^2$ square units. The larger rectangle has area ab square units while the smaller rectangle has area $b\,(a – b)$ square units. Thus, to obtain $(a – b)^2$ we can subtract the areas of the rectangles from the big square.

We obtain $(a – b)^2 = a^2 – ab – b(a – b) = a^2 – ab – ba + b^2$

$$= a^2 – 2ab + b^2.$$

Exercise Set 4.2

1. Factor completely:

   (i) $9x^2 + 24xy + 16y^2$

   (ii) $4s^2 + 20st + 25t^2$

(iii) $49x^2 + 28xy + 4y^2$

(iv) $64p^2 + \frac{32}{3}pq + \frac{4}{9}q^2$

*(v) $3a^2 + 4ab + \frac{4}{3}b^2$

*(vi) $\frac{9}{5}s^2 + 6sv + 5v^2$

(**Hint:** 2 was taken out as a common factor in Example 7. Is it possible to do something similar in Exercises (v) and (vi) above?)

2. Find the values of the following using the identity $(a - b)^2 = a^2 - 2ab + b^2$.

   (i) $(79)^2$ (ii) $(193)^2$ (iii) $(299)^2$

4.4. More Identities

What will happen if we want to find the square of the sum of three numbers a, b and c, that is, $(a + b + c)^2$?

Let us replace $b + c$ by d.

We already know, $(a + d)^2 = a^2 + 2ad + d^2$.

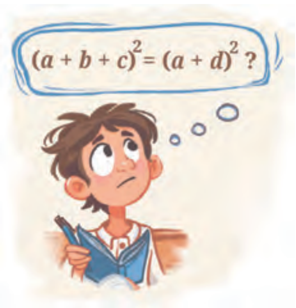
Thus, replacing d by $(b + c)$, we get $a^2 + 2ad + d^2 = a^2 + 2a(b + c) + (b + c)^2$.

So, $(a + b + c)^2 = a^2 + 2ab + 2ac + b^2 + 2bc + c^2$.

It may be more convenient to remember this as.

$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca.$

Let us interpret this geometrically by drawing a square of side $a + b + c$ as shown in Fig. 4.4.



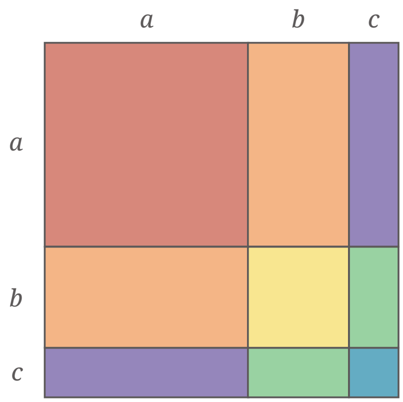


Fig. 4.4: A geometrical model representing the identity $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

Think and Reflect

Label the squares and rectangles in Fig. 4.4 so that it represents the identity $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$.

Example 9: Let us use this identity to find the square of a number, say 119:

$119^2 = (100 + 10 + 9)^2$

$= 100^2 + 10^2 + 9^2 + 2(100)(10) + 2(100)(9) + 2(10)(9)$

$= 10000 + 100 + 81 + 2000 + 1800 + 180 = 14161.$

So far we have verified the following three identities and used them for performing calculations and manipulating algebraic expressions:

1. $(a + b)^2 = a^2 + 2ab + b^2$
2. $(a - b)^2 = a^2 - 2ab + b^2$
3. $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

Exercise Set 4.3

1. Find the following squares using one of the above identities. Determine which of these identities will make these calculations easier.

   (i) 117^2 (ii) 78^2 (iii) 198^2

   (iv) 214^2 (v) 1104^2 (vi) 1120^2

2. Factor using suitable identities:

   (i) $16y^2 - 24y + 9$ (ii) $\frac{9}{4}s^2 + 6st + 4t^2$

   (iii) $\frac{m^2}{9} + \frac{mk}{3} + \frac{k^2}{4} + 3nk + 2mn + 9n^2$ (iv) $\frac{p^2}{16} - 2 + \frac{16}{p^2}$

   (v) $9a^2 + 4b^2 + c^2 - 12ab + 6ac - 4bc$

3. Expand the following using the identity

   $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$:

   (i) $(p + 3q + 7r)^2$ (ii) $(3x - 2y + 4z)^2$

4. Is this an identity?

   $(a + b - c)^2 + (a - b + c)^2 + (a - b - c)^2 = 2a^2 + 2b^2 + 2c^2$.

In Grade 8, you were introduced to yet another identity, $a^2 - b^2 = (a + b)(a - b)$.

This can be quite useful if it is rewritten as $a^2 = (a + b)(a - b) + b^2$.

Look at the following figure. Justify the identity $a^2 = (a + b)(a - b) + b^2$ for yourself.

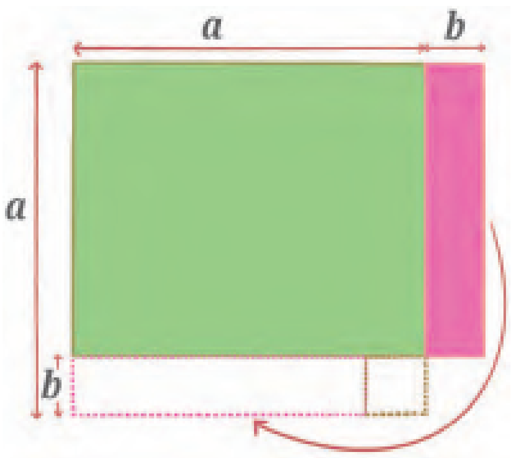


Fig. 4.5

In 750 CE, this identity was proposed by Śhrīdharāchārya as a method to quickly compute the squares of numbers. For example,

$$55^2 = (55 + 5)(55 - 5) + 5^2$$
$$= 60 \times 50 + 25$$
$$= 3000 + 25 = 3025.$$

Think and Reflect

1. Try to evaluate the following using a suitable identity:

   (i) 35^2 (ii) 65^2 (iii) 85^2 (iv) 105^2

   Do you observe any interesting pattern?

2. Observe the two rows of figures below. They represent an algebraic identity. Try to identify it.

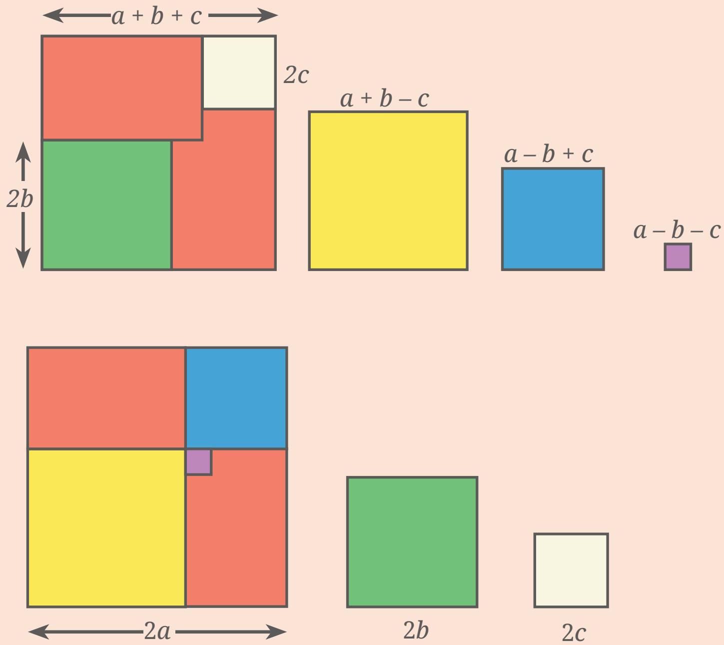


Fig. 4.6

4.5 Factorisation Using Algebra Tiles

Consider a rectangle with sides $x + 3$ and $x + 4$ units.

We know that the area of such a rectangle is $(x + 3)(x + 4)$ sq. units.

Using distributivity, we get

$$(x + 3)(x + 4) = x^2 + 3x + 4x + 12 = x^2 + 7x + 12.$$

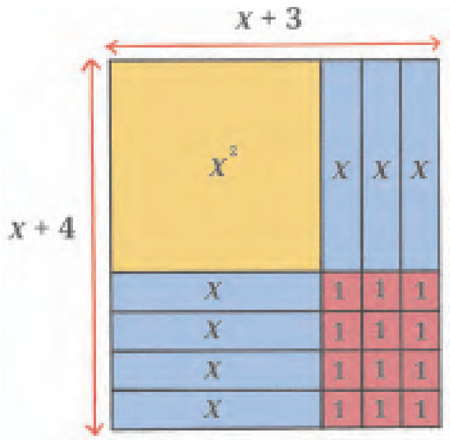


Fig. 4.7: Factorisation of $x^2 + 7x + 12$

Fig. 4.7 helps to visualise the product of $x + 3$ and $x + 4$ using algebra tiles. The expression $x + 3$ is represented using an x-tile and three unit tiles. Similarly $x + 4$ is represented using an x-tile and four unit tiles. The product of these two linear factors is shown in the inner rectangle comprising the x^2-tile, the 7 x-tiles, and 12 unit tiles.

Note that the $7x$ in $x^2 + 7x + 12$ has been split as $3x + 4x$. This is represented by the fact that three x-tiles are placed on the right side of the x^2-tile and four x-tiles are arranged below it. Also, the 12 unit tiles are arranged in a 3 by 4 array. Once the rectangular arrangement is formed, we observe that the dimensions of the rectangle are $x + 3$ and $x + 4$ units respectively.

Think and Reflect

Suppose $7x$ is split as $2x + 5x$; can a similar rectangular arrangement be formed? Consider other possibilities and check.

Fig. 4.7 helps us to visualise two algebraic identities:

(i) The linear expressions $x + 3$ and $x + 4$ can be multiplied to obtain the identity $(x + 3)(x + 4) = x^2 + 7x + 12$.

(ii) Also the expression $x^2 + 7x + 12$ can be factored into the linear factors $(x + 3)$ and $(x + 4)$, giving the same identity $x^2 + 7x + 12 = (x + 3)(x + 4)$.

Think and Reflect

Algebra tiles can be used to represent products and find factors.

1. Figure out the product of $x + 2$ and $x + 3$ using algebra tiles.
2. Lay out algebra tiles for $x^2 + 11x + 30$ in such a way that you will see its factors.

Think and Reflect

We have seen that $(x + 3)(x + 4) = x^2 + 7x + 12$.

Also $(x + 6)(x + 7) = x^2 + 13x + 42$.

Generalise the pattern to get an expression for $(x + a)(x + b)$.

Now consider the case where we have a rectangle of sidelengths $2x + 3$ and $3x + 1$, as shown in Fig. 4.8. What can you say about its area $(2x + 3)(3x + 1)$?

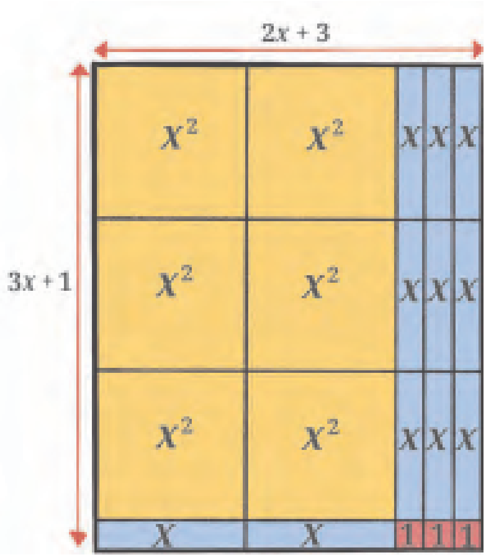


Fig. 4.8: Using algebra tiles to represent $(2x + 3) \times (3x + 1)$

Fill in the blanks with the appropriate expressions to make the equation true.

$$(px + a)(qx + b) = (\_\_\_\_)x^2 + (\_\_\_\_)x + \_\_\_\_ .$$

Also, verify your answer using the distributive property.

4.6 Factorisation Without Using Algebra Tiles

Consider the algebraic expression $x^2 + 7x + 12$ which was obtained by multiplying the linear terms $x + 3$ and $x + 4$. We also saw in Fig. 4.7 that $7x$ was split as $3x + 4x$ so that a rectangle could be formed using the algebra tiles. But how do we achieve this 'splitting of the x term' without using tiles?

Example 10: Let us begin with $x^2 + 7x + 12 = x^2 + (a + b)x + ab$.

Comparing the coefficients of the x-term and the constant terms on both sides of the equation, we get $a + b = 7$ and $ab = 12$. Note that this is only possible when $a = 3$ and $b = 4$, or $a = 4$ and $b = 3$. Thus, we choose one of these two possibilities and write the factors as

$(x + a)(x + b)$, that is, $(x + 3)(x + 4)$.

Example 11: Let us try to factor $x^2 + 11x + 30$ in a similar manner.

$$x^2 + 11x + 30 = x^2 + (a + b)x + ab.$$

This leads to $a + b = 11$ and $ab = 30$. We need to choose values of a and b appropriately, so that $a + b = 11$ and $ab = 30$ are both satisfied. What if we choose $a = 2$ and $b = 15$? Or $a = 3$ and $b = 10$? Clearly these will not work as $a + b$ is not equal to 11 even though $ab = 30$. So, we look at the factors of 30 and arrive at $a = 5$ and $b = 6$ or vice-versa. This way the x-term, $11x$, can be split as $5x + 6x$.

Thus, $x^2 + 11x + 30 = x^2 + (5 + 6)x + 30 = (x + 5)(x + 6)$.

Example 12: In order to factor $x^2 - 5x + 6$, we first note that the coefficient of x is negative. Once again comparing $x^2 - 5x + 6$ with $x^2 + (a + b)x + ab$, we get $a + b = -5$ and $ab = 6$. Note that these two equations can be satisfied together only when $a = -2$ and $b = -3$ or vice-versa.

Exercise Set 4.4

1. Fill in the blanks to complete the following identities:

   (i) $s^2 - 11s + 24 = (\_\_\_\_\_\_\_\_)(\_\_\_\_\_\_\_\_)$

   (ii) $(\_\_\_\_\_\_\_\_)(x + 1) = (3x^2 - 4x - 7)$

   (iii) $10x^2 - 11x - 6 = (2x - \_\_\_)(\_\_\_ + 2)$

   (iv) $6x^2 + 7x + 2 = (\_\_\_\_\_\_\_\_\_\_)(\_\_\_\_\_\_\_\_\_\_)$

2. Select and use the identity that will help you to find the following products without multiplying directly:

   (i) $(41)^2$ (ii) $(27)^2$

   (iii) (23×17) (iv) $(135)^2$

(v) $(97)^2$ (vi) (18×29)

(vii) (34×43) (viii) $(205)^2$

3. Factor the following:

(i) $9a^2 + b^2 + 4c^2 - 6ab + 12ac - 4bc$ (ii) $16s^2 + 25t^2 - 40st$

(iii) $r^2 - r - 42$ (iv) $49g^2 + 14gh + h^2$

(v) $64u^2 + 121v^2 + 4w^2 - 176uv - 32uw + 44vw$

Think and Reflect

James and Reshma were talking about algebraic identities they learnt in school.

James: $(a - b)^2 (a + b) = (a^2 - 2ab + b^2)(a + b)$

Reshma: I have a different idea. $(a - b)^2 (a + b) = (a - b) [(a - b) (a + b)]$

$= (a - b)(a^2 - b^2)$

I will find this product to get the answer.

According to you, who is correct and why?

Try to combine more such identities and find new results.

4.7 Finding New Identities

In this section we will play around with algebraic expressions to see if we can arrive at new identities.

What do you think $(a + b)^3$ will look like?

Let us find out the answer using the distributive property:

$$(a + b)^3 = (a + b)(a^2 + 2ab + b^2) = a^3 + 3a^2b + 3ab^2 + b^3.$$

Here we have found a new identity, namely,

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

Let us try to visualise this identity. We saw that a square of side $a + b$ can be divided into squares and rectangles. This led us to the identity, $(a + b)^2 = a^2 + 2ab + b^2$. What if we have a cube of edge $a + b$? Can we divide a cube of edge $(a + b)$ into smaller cubes and cuboids and represent this new identity?

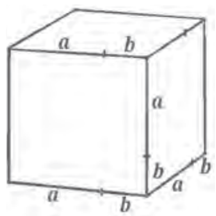


Fig. 4.9: A cube of edge a + b

We know that the volume of this cube is $(a + b)^3$. Let us split this cube into smaller cubes and cuboids.

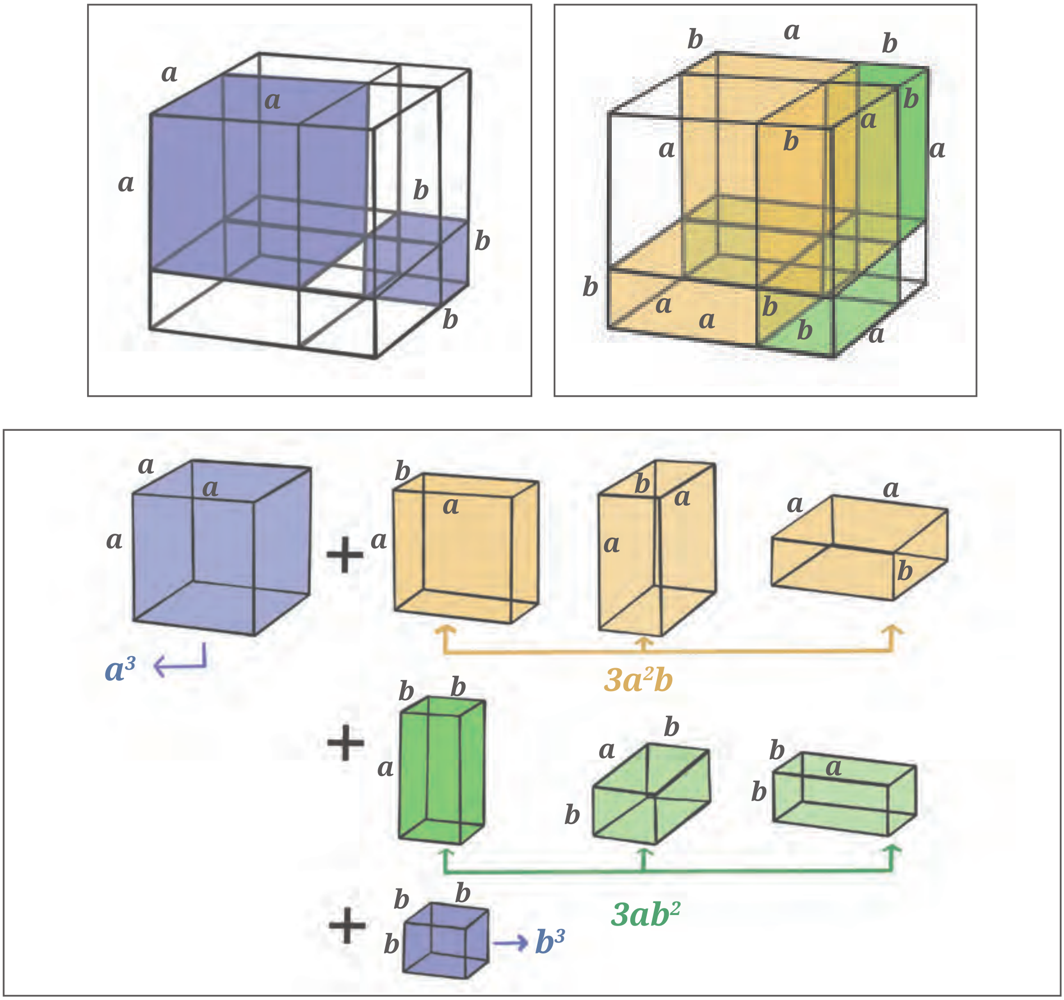


Fig. 4.10: Representation of the identity
$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

Notice that the larger cube can be split into two cubes and six cuboids. The cubes have volumes, a^3 and b^3 cubic units, respectively.

Amongst the other six cuboids, three have dimensions a units × a units × b units and other three have dimensions a units × b units × b units, and so their volumes are a^2b and ab^2 cubic units, respectively. Hence the total volume of the six cuboids is $3a^2b + 3ab^2$.

This gives us, $(a + b)^3 = (a + b)(a^2 + 2ab + b^2) = a^3 + 3a^2 b + 3ab^2 + b^3$.

Here we have found a new identity, namely,

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

What happens when we replace b with $-b$ in this new identity?

$$[a+(-b)]^3 = a^3 + 3a^2(-b) + 3a(-b)^2 + (-b)^3.$$

This leads to the identity $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$. Note that out of the four terms, two are positive and two are negative. They appear alternately in the expression.

Let us see how we can use these new identities.

Example 13: What is the side of the cube whose volume is $p^3 + 6p^2q + 12pq^2 + 8q^3$ cubic units?

Comparing $p^3 + 6p^2q + 12pq^2 + 8q^3$ with the right side of the identity $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, we may rewrite it as

$$(p)^3 + 3\,(p)^2(2q) + 3(p)(2q)^2 + (2q)^3$$

which is $(p + 2q)^3$, so that $a = p$ and $b = 2q$. Hence a side of the cube will be $p + 2q$ units.

Example 14: Now consider the expression

$$8n^3 - 60n^2m + 150nm^2 - 125m^3.$$

If you write it in the form $(a - b)^3$, what will be a and b?

Rewriting the expression $8n^3 - 60n^2m + 150nm^2 - 125m^3$ as $(2n)^3 - 3(2n)^2(5m) + 3(2n)(5m)^2 - (5m)^3$ and comparing it with $a^3 - 3a^2b + 3ab^2 - b^3 = (a-b)^3$, we get $(2n - 5m)^3$. Hence $a = 2n$ and $b = 5m$.

Now let us play with known identities to discover more identities. Try to multiply the following using the distributive property.

1. $(x - y)(x^2 + xy + y^2)$
2. $(x + y)(x^2 - xy + y^2)$

Let us work out the product of the first one.

$$(x-y)(x^2+xy+y^2) = x^3 + x^2y + xy^2 - x^2y - xy^2 - y^3$$

$$= x^3 + x^2y + xy^2 - x^2y - xy^2 - y^3 = x^3 - y^3.$$

Thus $(x-y)(x^2+xy+y^2) = x^3-y^3$. This is also an identity!

Verify this for yourself by choosing different values of x and y.

Predict what $(x+y)(x^2-xy+y^2)$ will be.

Think and Reflect

We already know that $x^2-y^2=(x-y)(x+y)$.

Further, we have verified that $x^3-y^3=(x-y)(x^2+xy+y^2)$.

Observe that $x-y$ is a common factor of x^2-y^2 and x^3-y^3.

Do you think $x-y$ is also a factor of x^4-y^4?

Note that $x^4-y^4=(x^2)^2-(y^2)^2=(x^2-y^2)(x^2+y^2)$.

Can you see how $x-y$ is a factor of x^4-y^4?

How about x^5-y^5? Does this also have $x-y$ as a factor?

Exploring further, let us multiply $(x+y+z)$ and $(x^2+y^2+z^2-xy-xz-yz)$.

$(x+y+z)(x^2+y^2+z^2-xy-xz-yz)$

$= (x^3+xy^2+xz^2-x^2y-x^2z-xyz)+(x^2y+y^3+yz^2-xy^2-xyz-y^2z)$

$\quad + (x^2z+y^2z+z^3-xyz-xz^2-yz^2)$

$= (x^3+\cancel{xy^2}+\cancel{xz^2}-\cancel{x^2y}-\cancel{x^2z}-xyz)+(\cancel{x^2y}+y^3+\cancel{yz^2}-\cancel{xy^2}-xyz-\cancel{y^2z})$

$\quad + (\cancel{x^2z}+\cancel{y^2z}+z^3-xyz-\cancel{xz^2}-\cancel{yz^2})$

$= x^3+y^3+z^3-3xyz.$

So, $(x+y+z)(x^2+y^2+z^2-xy-xz-yz) = x^3+y^3+z^3-3xyz$.

This is yet another identity. Let us explore some of its applications.

Example 15: The sum of three numbers is 10 and their product is 25. The sum of their squares is 38. Try to use the previous identity to find the sum of the cubes of these three numbers.

Let the three numbers be x, y and z, respectively. According to the problem $x+y+z=10$, $xyz=25$ and $x^2+y^2+z^2=38$.

Substituting in the identity

$(x+y+z)(x^2+y^2+z^2-xy-xz-yz) = x^3+y^3+z^3-3xyz$, we get

$(10)(38-xy-xz-yz) = x^3+y^3+z^3-3\,(25)$.

Thus, $x^3+y^3+z^3 = 380-10\,(xy+xz+yz)+75 = 455-10\,(xy+xz+yz)$.

To find $(xy + xz + yz)$, we need to use the identity
$(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$.

We get $100 = 38 + 2\,(xy + xz + yz)$. Thus $xy + xz + yz = 31$.

Now, substituting this into our earlier equation, we obtain

$x^3 + y^3 + z^3 = 455 - 10\,(xy + xz + yz) = 455 - 10\,(31) = 455 - 310 = 145$.

4.8 Simplifying Rational Expressions

In the earlier sections of this chapter, we have learnt how to factorise algebraic expressions. Let us see how we can simplify some rational algebraic expressions using factorisation.

Example 16: Simplify the rational expression $\dfrac{x^2 - 7x + 12}{5x^2 + 5x - 100}$, assuming that $5x^2 + 5x - 100 \neq 0$.

In order to simplify this rational expression, we need to cancel the common factors between the expressions in the numerator and denominator.

We will need to use algebraic identities to simplify this rational expression.

First, let us look at the numerator $x^2 - 7x + 12$. We need to find a and b such that $a + b = -7$ and $ab = 12$. Can you think of two such numbers? They are -3 and -4. Factor the expression, we get $x^2 - 7x + 12 = (x - 3)(x - 4)$.

Now let us look at the denominator $5x^2 + 5x - 100$. We can see that all the three terms are multiples of 5 so we can take 5 as a common factor and simplify the expression: $5x^2 + 5x - 100 = 5(x^2 + x - 20)$. Now we need to find a and b such that, $a + b = 1$ and $ab = -20$. Try to think of two such numbers. They are 5 and -4. Thus, we have $5x^2 + 5x - 100 = 5\,(x - 4)(x + 5)$.

Substituting the factored expressions in the numerator and denominator, we arrive at

$$\frac{x^2 - 7x + 12}{5x^2 + 5x - 100} = \frac{x^2 - 7x + 12}{5(x^2 + x - 20)} = \frac{(x-4)(x-3)}{5(x-4)(x+5)}.$$

Thus, the common factor $x - 4$ can be cancelled as it is not equal to 0. We know this because $5x^2 + 5x - 100 \neq 0$. Thus,

$$\frac{x^2 - 7x + 12}{5x^2 + 5x - 100} = \frac{x-3}{5(x+5)}.$$

Think and Reflect

Try to simplify the following rational expression:

$$\frac{36s^2 - 12st + t^2}{t^2 + 2ts - 48s^2} = \frac{(6s-t)^2}{(\_\_+\_\_)(\_\_+\_\_)}.$$

(**Hint:** Factor $t^2 + 2ts - 48s^2$ and simplify the rational expressions assuming that $t^2 + 2ts - 48s^2 \neq 0$).

Exercise Set 4.5

1. Simplify the following rational expressions assuming that the expressions in the denominators are not equal to zero:

(i) $\dfrac{3p^2 - 3pq - 18q^2}{p^2 + 3pq - 10q^2}$ (ii) $\dfrac{n^3 - 3n^2m + 3nm^2 - m^3}{5m^2 - 10mn + 5n^2}$

(iii) $\dfrac{w^3 - v^3 + x^3 + 3wvx}{w^2 + v^2 + x^2 - 2wv - 2vx + 2wx}$ (iv) $\dfrac{4y^2 - 20yz + 25z^2}{\left(25z^2 - 4y^2\right)}$

(v) $\dfrac{\left(x^2 + x - 6\right)\left(x^2 - 7x + 12\right)}{(x^2 - 6x + 8)\left(x^2 - 9\right)}$ (vi) $\dfrac{p^4 - 16}{p^2 - 4p + 4}$

Example 17: Saira has arranged a square of side x units, 8 rectangular strips of sides x units and width 1 unit, and 15 squares of side 1 unit to form a bigger rectangle. Find the length and breadth of the rectangle in terms of x.

Area covered by a square of side x units = x^2 sq. units

Area covered by rectangle of sides x units and 1 unit = x sq. units

Total area covered by 8 such rectangles = $8x$ sq. units

Area covered by a square of side 1 unit = 1 sq. units

Total area covered by 15 such squares = 15 sq. units

Total area covered by all the squares and rectangles

= $x^2 + 8x + 15$ sq. units.

Area of Saira's rectangle = $x^2 + 8x + 15$ sq. units.

We can factor $x^2 + 8x + 15$ to obtain the dimensions of the rectangle prepared by Saira.

For factoring $x^2 + 8x + 15$, we need a and b such that $a + b = 8$ and $ab = 15$

So, $a = 3$ and $b = 5$ is one possibility. The possible length and breadth of Saira's rectangle in terms of x are: length = $x + 5$ units and breadth = $x + 3$ units.

Now draw Saira's rectangle using these pieces.

Example 18: A rectangular pool is such that its breadth is 4 metres less than its length and its area is 96 sq. metres. Find the length and breadth of the pool.

Let the length of the pool be x metres. Then the breadth of the pool is $x - 4$ metres.

Since the area is given to be 96 sq. metres, we get $x(x - 4) = 96$.

$$x^2 - 4x = 96 \text{ or } x^2 - 4x - 96 = 0.$$

We choose appropriate factors of -96 to split the term $-4x$. We know that, $(-12) \times 8 = 96$ and $(-12) + 8 = -4$. Hence $-4x$ can be written as $-12x + 8x$.

$$x^2 - 4x - 96 = x^2 - 12x + 8x - 96 = x(x - 12) + 8(x - 12) = (x - 12)(x + 8) = 0.$$

Thus, $x^2 - 4x - 96 = 0$ implies that $(x - 12)(x + 8) = 0$.

This means either $x - 12 = 0$ or $x + 8 = 0$.

That is, $x = 12$ or $x = -8$. Since x is the length of the pool, it cannot be negative. Therefore, we ignore $x = -8$ and $x = 12$ metres must be the length of the pool. The breadth of the pool = $x - 4 = 12 - 4 = 8$ metres.

End-of-Chapter Exercises

1. Use suitable identities to find the following products:

(i) $(-3x + 4)^2$ (ii) $(2s + 7)(2s - 7)$

(iii) $\left(p^2 + \frac{1}{2}\right)\left(p^2 - \frac{1}{2}\right)$ (iv) $(2n + 7)(2n - 7)$

(v) $(s - 2t)(s^2 + 2st + 4t^2)$ (vi) $\left(\frac{1}{2r} - 4r\right)^2$

(vii) $(-3m + 4k - l)^2$ (viii) $\left(x - \frac{1}{3}y\right)^3$

(ix) $\left(\frac{7}{2}k - \frac{2}{3}m\right)^3$

2. Find the values using suitable identities:
   (i) 17×21 (ii) 104×96
   (iii) 24×16 (iv) 147^3
   (v) 199^3 (vi) 127^3
   (vii) $(-107)^3$ (viii) $(-299)^3$

3. Factor the following algebraic expressions:
   (i) $4y^2 + 1 + \frac{1}{16y^2}$ (ii) $9m^2 - \frac{1}{25n^2}$
   (iii) $27b^3 - \frac{1}{64b^3}$ (iv) $x^2 + \frac{5x}{6} + \frac{1}{6}$
   (v) $27u^3 - \frac{1}{125} - \frac{27u^2}{5} + \frac{9u}{25}$ (vi) $64y^3 + \frac{1}{125}z^3$
   (vii) $p^3 + 27q^3 + r^3 - 9pqr$ (viii) $9m^2 - 12m + 4$
   (ix) $9x^3 - \frac{8}{3}y^3 + \frac{z^3}{3} + 6xyz$
   (x) $4x^2 + 9y^2 + 36z^2 + 12xz + 36yz + 24xy$
   (xi) $27u^3 - \frac{1}{216} - \frac{9u^2}{2} + \frac{u}{4}$

4. Simplify the following:
   (i) $\frac{4x^2 + 4x + 1}{4x^2 - 1}$ (ii) $\frac{9\left(3a^3 - 24b^3\right)}{9a^2 - 36b^2}$ (iii) $\frac{s^3 + 125t^3}{s^2 - 2st - 35t^2}$

Note: Assume that the denominators are not equal to 0.

5. Find possible expressions for the length and breadth of each of the following rectangles whose areas are given by the following expressions in square units.
   (i) $25a^2 - 30ab + 9b^2$ (ii) $36s^2 - 49t^2$

6. Find possible expressions for the length, breadth, and heights of each of the following cuboids whose volumes are given by the following expressions in cubic units.
   (i) $6a^2 - 24b^2$ (ii) $3ps^2 - 15ps + 12p$

7. The village playground is shaped as a square of side 40 metres. A path of width s metres is created around the playground for people to walk. Find an expression for the area of the path in terms of s.

8. If a number plus its reciprocal equals $\frac{10}{3}$, find the number.

9. A rectangular pool has area $2x^2 + 7x + 3$ square *hastas*. If its width is $2x + 1$ *hastas*, find its length. *Hasta* was a unit used to measure length.

*10. If both $x - 2$ and $x - \frac{1}{2}$ are factors of $px^2 + 5x + r$, show that $p = r$.

*11. If $a + b + c = 5$ and $ab + bc + ca = 10$, then prove that $a^3 + b^3 + c^3 - 3abc = -25$.

*12. By factoring the expression, check that $n^3 - n$ is always divisible by 6 for all natural numbers n. Give reasons.

*13. Find the value of

(i) $x^3 + y^3 - 12xy + 64$, when $x + y = -4$

(ii) $x^3 - 8y^3 - 36xy - 216$, when $x = 2y + 6$

Chapter Summary

- **Identities** are equations that are true for all values of the variables.
- One of the ways to visualise identities is using geometrical models or algebra tiles.
- Identities can also be used to factor algebraic expressions.
- Factorisation of quadratic expressions may be visualised by means of algebra tiles.
- Identities can also be used to simplify calculations such as squaring numbers or evaluating products of numbers.
- Rational algebraic expressions may be simplified by factorisation and removing the common factors in the numerator and denominator, provided such a factor exists and it is not equal to zero.

- We have studied the following identities in this chapter:
  - $(x + y)^2 = x^2 + 2xy + y^2$
  - $(x - y)^2 = x^2 - 2xy + y^2$
  - $(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$
  - $(x + y)(x - y) = x^2 - y^2$
  - $(x + a)(x + b) = x^2 + (a + b)x + ab$
  - $(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd$
  - $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$
  - $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$
  - $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
  - $(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$
  - $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - xz - yz)$

5 I'm Up and Down, and Round and Round

Humanity has always been fascinated by the shapes of the things around them. In some early cave paintings, the sun is depicted as a circle. In the cave paintings of Gudahandi in Odisha, one sees numerous geometric patterns including triangles, squares, circles, and ovals. These shapes were likely inspired by what humans saw in nature.

Can you recognise the origin of the shapes in Fig. 5.1?

Fig. 5.1

Circles form when raindrops fall on water. The cross-section of a plant stem and the inflorescence of a sunflower are also circular in shape. The full moon and sun also look circular.

Moon

Sun (during a total solar eclipse)

Fig 5.2

What properties are common to all circles, big and small? You have studied one such property in Grade 7. Humans must have noticed it after observing many circular patterns in nature. Every circle has a centre. All points on the circle are at equal distance from the centre. We turn this observation around and make it the definition of a circle.

Activity: List some objects from nature that resemble a circle.

Think and Reflect

Jamuna has a circular piece of paper. She is trying to locate its centre. Amina gives her a suggestion. She follows the instructions and is thrilled to find that it works. Can you guess what Amina told her?

5.1 Definitions

When we talk of mathematical shapes such as circles, triangles and squares, we always assume that the figures are drawn on a piece of paper — a two-dimensional plane.

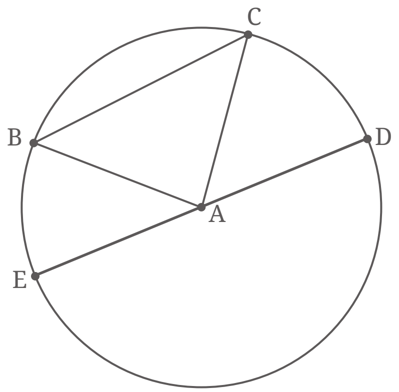


Fig. 5.3: Circle, Centre A, Chord BC

A **circle** is the set of all points on the plane that are equidistant from a given point on that plane. The set of points that satisfy a given condition is also called the **locus** of points that satisfy the condition. Using this term, a circle can also be described as the locus of points that are equidistant from a given point. The given point is the **centre** of the circle. The distance from the centre to any point on the circle is the **radius** of the circle.

In Fig 5.3, A is the centre of the circle. All points on the plane at a distance equal to the length of AB from A form a circle with centre A and radius equal to the length of AB.

Let B and C be two points on the circle. The line segment BC is called a **chord** of the circle. The angle subtended by the chord BC at the centre is angle BAC. A chord passing through the centre of a circle is called a **diameter**.

5.2 Symmetries of a Circle

What makes circles so appealing is that they are perfectly symmetrical. Say you are looking at a wheel of a vehicle. You see a point of the wheel touching the ground. When you look at the wheel again after some time, you again see a point of the wheel touching the ground. Can you tell if the two points are the same point? There is no way you can tell; a rotating wheel looks the same at all times! We say the circle has complete rotational symmetry: rotate it by any angle and it looks exactly the same.

Draw a circle on the paper and cut along the circle. Fold the circular paper so that the boundaries overlap, then open it. You see a crease; it is a line of reflection symmetry of the circle. Does this line pass through the centre of the circle? It does. It is a **diameter** of the circle. All diameters are lines of reflection symmetry.

Think and Reflect

1. What are the rotational symmetries of a square? How many lines of reflection symmetry does it have? What about a regular pentagon? A regular hexagon?
2. What is the length of the longest chord in a circle of radius 5 units? Is there a smallest chord?
3. The locus of points at a given distance from a given point is a circle. What can we say about the locus of points equidistant from two given points?

   (**Hint:** We know that any point that is equidistant from two given points A and B lies on the perpendicular bisector of AB. Does this make the perpendicular bisector the locus? For this, we have to show that all the points on the perpendicular bisector are equidistant from A and B.)

5.3 How Many Circles?

Now that we have defined a circle and listed some of its properties, let us ask this question: Given two points A and B on a plane, how many circles pass through A and B?

If a circle passes through A and B, it has a centre, say O. The lengths OA and OB are equal. Is there another point with this property? Yes, the midpoint of segment AB. With the midpoint as centre, a circle passing through A and B can be drawn. Its radius is half the length of AB, and AB is a diameter.

Are there more circles passing through both A and B? Clearly, any point equidistant from A and B can be the centre of a circle passing through A and B. Do you know of such points? We have seen that the perpendicular bisector of the line segment AB is the locus of points equidistant from A and B. Every point on the perpendicular bisector is equidistant from A and B, and every point that is equidistant from A and B is on the perpendicular bisector! So, the centres of all circles through A and B lie on the perpendicular bisector of AB.

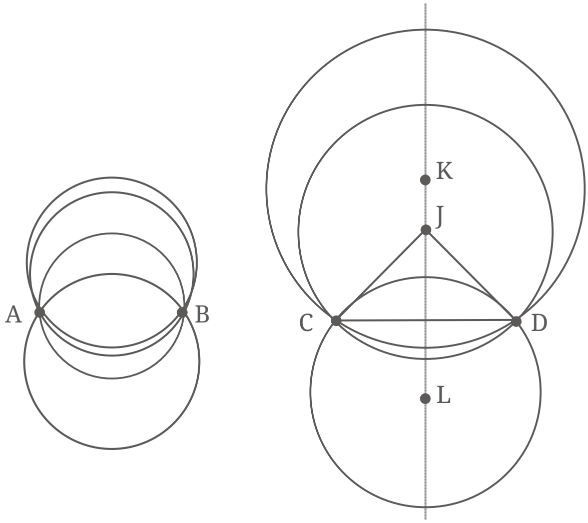


Fig. 5.4: Circles through two points

In Fig. 5.4, we see circles through points A and B, and through C and D, and the perpendicular bisector of CD. Every point on the perpendicular bisector is the centre of a circle passing through C and D. So we have circles with centres K, J and L containing points C and D.

Think and Reflect

1. How many circles pass through two points on a plane?
2. Are there circles of all possible radii passing through A and B? What is the radius of the smallest circle passing through A and B? What is the radius of the largest circle passing through A and B?
3. As you move away from segment AB along its perpendicular bisector, do the radii of the circles containing A and B increase or decrease?
4. As you go along the perpendicular bisector, will the circle drawn from that point through A and B appear more curved or less curved?
5. You are given two points A and B on a plane. How many squares can you draw on the same plane with A and B on the boundary? How many squares can you draw on the plane with A and B as the corners of the square?

It is natural to ask: How many circles can you draw through three distinct points A, B and C on a plane? Is there always at least one such circle? Not necessarily! What if A, B and C lie on a straight line, i.e., are **collinear**? Can you explain why, in this case, there is no circle through A, B and C?

Let us assume that A, B and C are not collinear. Is there always a circle passing through A, B and C? Can there be more than one circle through A, B and C?

Theorem 1: *There is a unique circle passing through three non-collinear points.*

Why is this true? If there is a circle that passes through non-collinear A, B and C, the circle must have a centre. Let's call that centre O (we will discover O soon).

Since OA = OB, we know that O lies on the perpendicular bisector of AB.

Since OA = OC, we know that O lies on the perpendicular bisector of AC as well.

But A, B and C are **not** collinear. So, the perpendicular bisectors of AB and AC will intersect at a unique point, since two intersecting lines meet at exactly one point on the plane. That point must be O.

Using O as the centre we can draw a circle with radius equal to the length of OA. That will pass through A, B and C. This explains why the statement is true.

Now three non-collinear points also determine a triangle. What we have described above is the construction of a circle that passes through the vertices of the triangle.

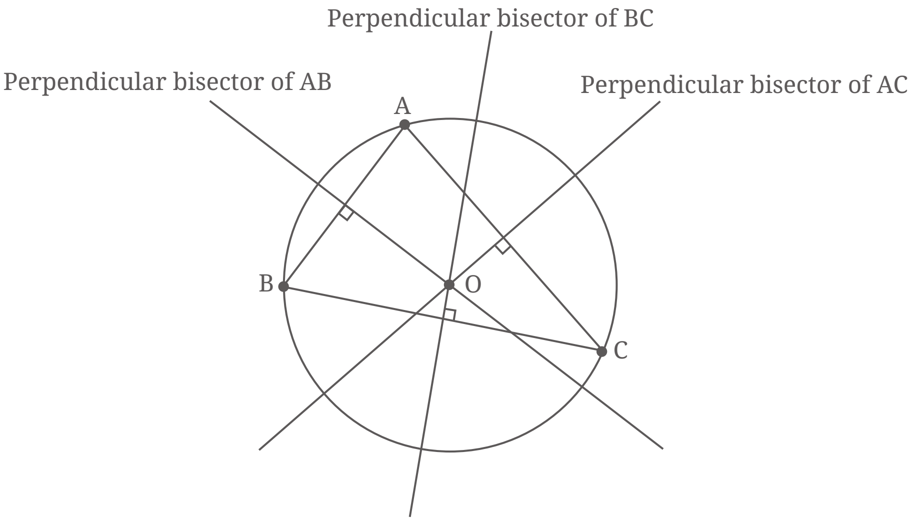


Fig. 5.5: The circumcircle of triangle ABC

The centre O of the circle that passes through the vertices A, B and C of ΔABC is called the **circumcentre** of ΔABC (Fig. 5.5). The circle is called the **circumcircle** and is said to circumscribe the triangle. Conversely, the triangle is said to be inscribed in the circle. Since, the intersection of the perpendicular bisectors of AB and BC is a single point, there is just one such circle for a given triangle. Further, for an acute-angled triangle, the circumcentre lies inside the triangle (see Fig. 5.5). For an obtuse-angled triangle, the circumcentre lies outside the triangle (Fig. 5.6). And for a right-angled triangle, the circumcentre lies at the midpoint of the hypotenuse (Fig. 5.7).

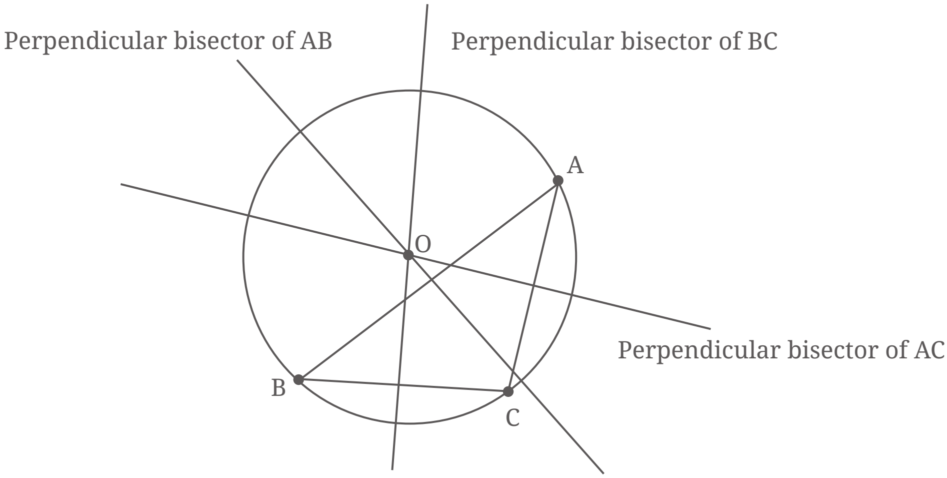


Fig. 5.6: Obtuse-angled triangle: Circumcentre O is outside the triangle

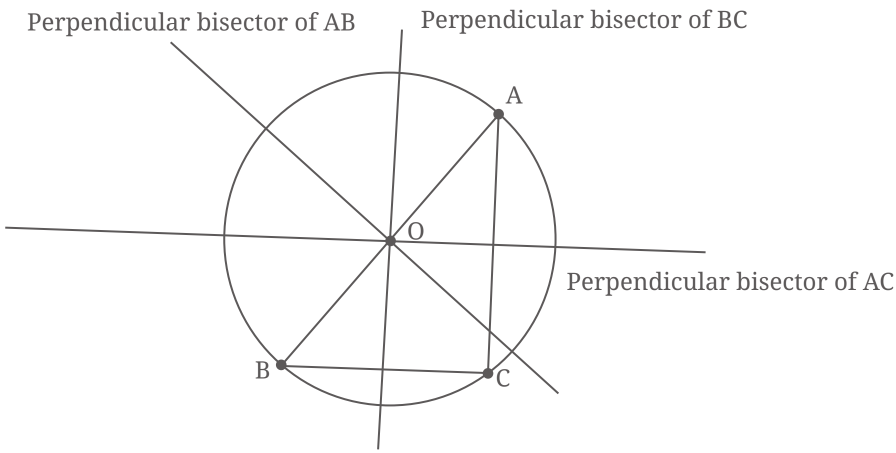


Fig. 5.7: Right-angled triangle: Circumcentre O is at the midpoint of the hypotenuse

Exercise Set 5.1

1. Draw ΔABC with AB = 5 cm, ∠A = 70° and ∠B = 60°. Draw the circumcircle of ΔABC. Is the centre inside or outside the triangle?
2. Draw ΔABC with AB = 5 cm, ∠A = 100°, AC = 4 cm. Draw the circumcircle of ΔABC. Is the centre inside or outside the triangle?
3. Draw ΔABC, with AB = 6 cm, BC = 7 cm and CA = 7 cm. Draw the circumcircle of ΔABC. Let the circumcentre be O. Measure OA, OB, OC.
4. What is the least possible radius of a circle through two points A and B?

Think, Draw and Infer

1. A, B and C are three collinear points. Can you find a point P such that PA = PB = PC? What can you say about the perpendicular bisectors of AB and BC? Draw and check. Can you show that for three collinear points A, B and C, the perpendicular bisector of AB and BC are parallel? Is it possible for a circle to pass through collinear points? Can you draw a line that cuts a given circle in three distinct points?
2. The circumcircle of a given ΔABC is drawn. Can there be other triangles congruent to ΔABC that share the same circumcircle?

5.4 Chords and the Angles They Subtend

You may want to know when 4 points lie on the same circle. That will have to wait till the end of this chapter. We need to do some more work to get there.

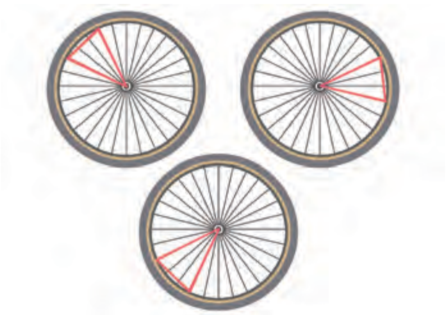


Fig. 5.8: Chords and radii

Tie a thread at two points on a wheel; pull it tight (Fig. 5.8). The thread can be thought of as a chord of the wheel (a circle). Imagine joining the end points of the thread to the centre. The thread subtends an angle at the centre. Now, rotate the wheel. The thread rotates around the centre along with the wheel.

In the new position, the thread is another chord on the circle. What is common to the first and second chords? They have the same length, equal to the length of the thread. What about the angle subtended by the second chord at the centre? It must surely be the same as the angle subtended by the first chord at the centre. We now explain why this is true.

Theorem 2: *Equal chords of a circle subtend equal angles at the centre of the circle.*

What does the theorem say? It talks of two chords of the circle with equal length. In geometry, it always helps to draw a figure! (See Fig. 5.9). AB and DE are chords of the same length. We need to explain why ∠ACB is equal to ∠DCE. What are we given? AB = DE.

To explain why two angles in two different triangles are equal, we use congruence of triangles. If the angles which we want to show equal are corresponding angles, then our task would be done!

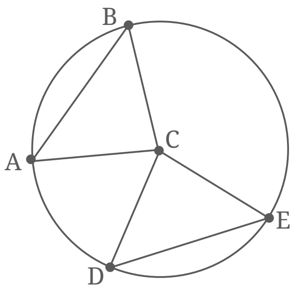


Fig. 5.9

Given: AB = DE.

To show: ∠ACB = ∠DCE.

Why is this true? First, we see that CA = CB = r, the radius of the circle.

Also, CD = CE = r. So, CA = CD and CB = CE.

But AB = DE is given! By the SSS congruence, ΔCAB is congruent to ΔCDE. So, ∠ACB is equal to ∠DCE, i.e., the angles subtended at the centre are equal.

Now suppose we have two chords that subtend equal angles at the centre. Are the chords equal? What does our intuition tell us? Let the chords be AB and DE. Let us use our imagination. Assume that B, A, E, and D are located clockwise on the circle as shown in Fig 5.10.

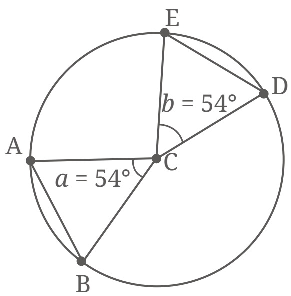


Fig. 5.10

Imagine you are standing at the centre C of a circle. Stretch your arms so that they are at a certain angle. Let your arms intersect the circle at A, B. The distance between your arm ends must be the length of chord AB. Now let the floor rotate magically. You start rotating (say clockwise) about C. Don't change the angle your arms make! At some point your left arm meets point E.

Where will your right hand be? It should be at D, should it not? That is because we assumed that angles ACB and DCF are equal, and you have not moved your hands. The distance between the ends of your arms must be the length of DE as well. Let us now explain why this is true.

Theorem 3: *Chords of a circle that subtend equal angles at the centre are equal.*

Let us draw the figure, as in Fig. 5.11.

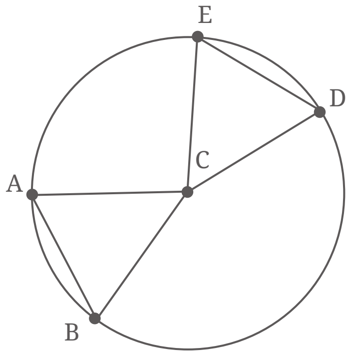


Fig. 5.11

Given: $\angle ACB = \angle DCE$.

To show: AB = ED.

Why is this true? We see that AC = BC = the radius of the circle. Likewise, EC = DC = the radius. So, AC = DC and BC = EC.

Since, $\angle ACB = \angle DCE$, by the SAS congruence, $\Delta ACB \cong \Delta DCE$. Hence, AB = ED.

Exercise Set 5.2

1. Show that the triangle formed by a chord and the centre of the circle is isosceles.
2. Show that if two such isosceles triangles (occurring in the previous question) have equal base length, they are congruent to each other.

5.5 Midpoints and Perpendicular Bisectors of Chords

We will explore more properties of chords. Do we get something special when we draw a line segment from the centre of a circle to the midpoint of a chord? What if we draw the perpendicular bisector of a chord? Does it pass through a special point?

Theorem 4: *The line joining the centre of a circle and the midpoint of a chord of the circle is perpendicular to the chord.*

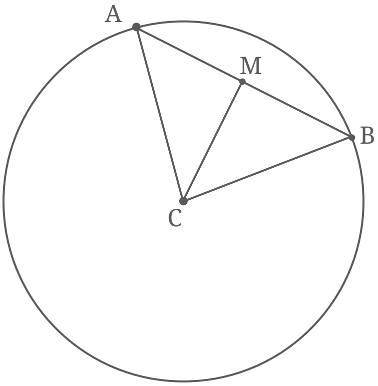


Fig. 5.12

Let us see why this is true. Draw a circle with centre C (Fig. 5.12). Draw any chord AB. Let M be the midpoint of AB. We must show that CM is perpendicular to AB.

Explanation: CAB is an isosceles triangle. AB is the base and CA = CB. So, ∠A = ∠B.

M is the midpoint of AB. So, AM = BM.

By the SAS congruence, Δ CMA is congruent to Δ CMB. So, ∠CMB = ∠CMA.

But ∠CMB + ∠CMA = 180° (angles on a line).

So, both angles are 90°. That is, CM is perpendicular to AB.

Exercise Set 5.3

1. Can you explain why the converse to Theorem 4 is true, i.e., why does the perpendicular from the centre of a circle to a chord of the circle bisect the chord?

   (**Hint:** Use Fig. 5.12. You are told that ∠CMA = ∠CMB = 90°. You need to show that AM = BM.)

2. An isosceles triangle ABC is inscribed in a circle, with AB = AC. Show that the altitude from A to BC passes through the centre of the circle.

3. Two parallel chords of lengths 6 cm and 8 cm are on opposite sides of the centre of a circle. If the radius of the circle is 5 cm, find the distance between the midpoints of the chords.

From Question 1 above, we have the following result.

Theorem 5: *The perpendicular from the centre of a circle to a chord of the circle bisects the chord.*

5.6 Distance of Chords from the Centre

Activity: Take a paper circle. Fold the circle from the boundary, inwards. Open the fold. The crease is now a chord (see Fig. 5.13B). Now fold the paper again, so that the end points of the chord meet. Open the fold (see Fig. 5.13C).

Fig. 5.13A

Fig. 5.13B

Fig. 5.13C

Measure the lengths of the parts into which the chord is divided. The chord gets bisected where the folds intersect. Measure the angle between the creases. The crease of the second fold is along the perpendicular from the centre to the chord. Measure the distance from the centre to the midpoint of the chord. It is the distance from the centre to the chord.

Now draw another chord of the same length. How will you do this? We will let you figure this out yourself. Join the centre to the midpoint of the new chord and measure its length. Is it the same as distance from the centre to the first chord?

Now take a tracing paper and draw a circle on it with the same radius. Place the circle on the tracing paper on top of the circle paper so that the circles overlap. Trace the chord on the circle paper, and the perpendicular to it from its centre, onto the tracing paper circle. Now rotate the tracing paper. You will see the chord through the tracing paper—it looks like a different chord of the same length on the circle paper.

The circle is a symmetric figure. So, rotating the circle around its centre does not change the circle. As we rotate the circle, our chosen chord will also rotate, to give a new chord of the same length. Along with it, the perpendicular from the centre to the chord also moves! Equal chords appear to be equidistant from the centre.

Is this a totally convincing explanation? No, we have only given examples where our guess holds true. A statement may be true on a large number of examples, but it does not mean the statement is true in general. So let us explain why the statement is true.

Theorem 6: *Chords of a circle having the same length are all at the same distance from the centre of the circle.*

Draw the figure (Fig. 5.14).

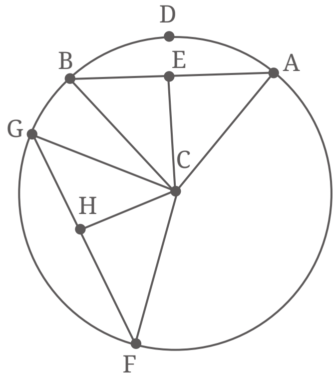


Fig. 5.14

Given: Circle with centre C; AB = FG; E, H are midpoints of AB, FG respectively.

To show: CE = CH.

Why is this true? From Theorem 5, CE and CH are the perpendiculars from C to the chords AB and FG respectively.

So, CE and CH are the distances from the centre C to the chords AB and FG respectively.

We shall explain this result in two different ways.

(1) Triangles CAB and CFG are congruent.

CA = CF (both are equal to the radius of the circle). Likewise, CB = CG.

We are given that AB = FG. So, by the SSS congruence, $\Delta CAB \cong \Delta CFG$.

Hence the altitudes of the congruent triangles are congruent. CE = CH. Therefore, the chords AB and FG are equidistant from the centre C.

(2) Consider triangles CEA and CHF.

From Theorem 5, the perpendicular from the centre bisects the chord.

Hence, E and H are the midpoints of the chords AB and FG respectively.

AE = FH (since AB = FG, and E and H are midpoints of AB and FG).

∠CEA = ∠CHF = 90°.

CF = CA, as they are radii of the circle.

By the RHS congruence, ΔCEA ≅ ΔCHF.

So, CE = CH. Thus the chords of equal length are equidistant from the centre.

Exercise Set 5.4

1. Use the Baudhāyana–Pythagoras theorem to show why Theorem 6 must be true.

2. Consider Fig. 5.15. If CE is perpendicular to AB, CH is perpendicular to GH, and CE = CH, show that AB = GF.

3. Solve the previous question using the Baudhāyana–Pythagoras theorem.

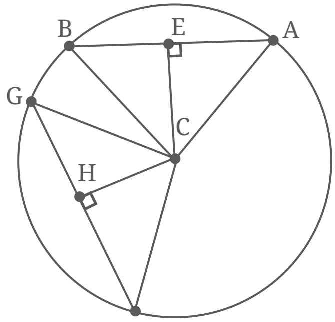


Fig. 5.15

Question 2 in the exercise above establishes the following result.

Theorem 7: *Chords of a circle that are equidistant from the centre have equal length.*

5.6.1 Which of the two unequal chords is farther from the centre?

You have two chords on a circle. One is longer than the other. Which chord is closer to the centre? Can you guess?

Activity: Draw a circle. Draw chords of various lengths. Drop a perpendicular to each chord from the centre. Record the length of the chord and its distance from the centre in a table.

Table 1

Length of Chord			
Distance from Centre			

What do you observe? The longer the chord, the closer it is to the centre. Let us try to understand why this is true.

Theorem 8: *Let AB and DE be two chords of a circle with centre C. Suppose AB > DE. Then the distance from C to AB is less than the distance from C to DE.*

As always, we draw the figure. Remember: by 'distance from C to AB' we mean the perpendicular distance. So, we drop perpendiculars CF and CG from C to AB and DE, respectively.

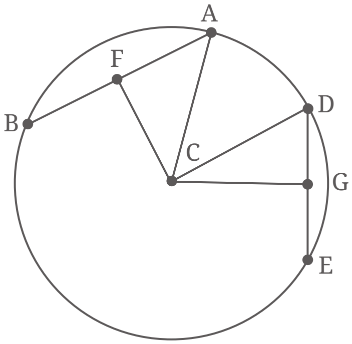


Fig. 5.16

Given: AB > DE.

To show that: CF < CG.

Why is this true? In Fig. 5.16, both AC and CD are radii. So, AC = CD. By the Baudhāyana–Pythagoras Theorem, $CD^2 = CG^2 + GD^2$ and $AC^2 = CF^2 + AF^2$.

So, $CF^2 + AF^2 = CG^2 + GD^2$.

Now AB is greater than DE. So AF is greater than GD (because F and G are the midpoints of AB and DE).

Since $AF^2 > GD^2$ we have $CF^2 < CG^2$.

So, CF < CG.

Comment: The chord nearest to the centre is the chord containing the centre. Its distance to C is zero! So the diameter is the greatest chord.

If one pushes the chord away from the centre, at some stage the chord becomes a single point, its length is zero, and the distance of C from this chord is the radius.

Exercise Set 5.5

1. Find the length of the chord of a circle where the radius is 7 cm and perpendicular distance is 6 cm.
2. Explain why the following statement is true: If the perpendicular distance of a chord from the centre is d and the radius is r, then the chord length is $2\sqrt{r^2 - d^2}$.

*3. In a circle, if the distance of chord AB from the centre is twice the distance of another chord CD from the centre, then can we conclude that CD = 2 AB? Give reasons for your answer.

5.7 Angles Subtended by an Arc

An **arc** of a circle is a connected portion of the circle. It is defined by two points on the circle, called the **end points** of the arc, and the curve connecting them along the circle's edge.

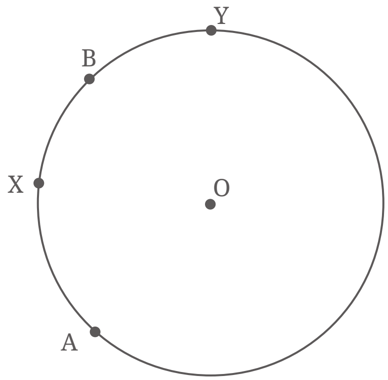


Fig. 5.17: Major arc AYB and minor arc AXB

Look at Fig. 5.17. A and B are two points on the circle. There are two ways of going from A to B. One goes via point X and one goes via point Y. The bigger of the two is called the major arc, and the smaller of the two is called the minor arc.

We define the **angle subtended** by the arc AB at the centre to be the measure of the angle AOB, as we sweep along the arc—so we move from OA to OB along the arc and measure the angle swept. The minor arc subtends the ∠AOB—here you move from OA to OB via X. The major arc subtends the angle we get as we move from OA to OB via Y (see Fig. 5.18).

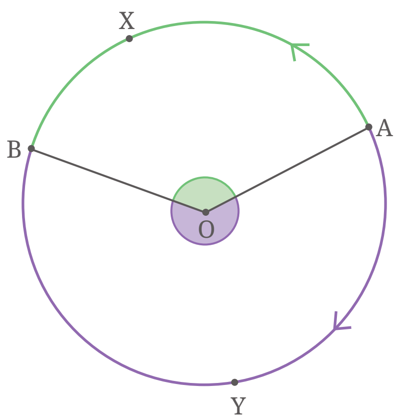


Fig. 5.18

Exercise: A circle with centre O is drawn, and A, B, C, D are points on the circle (see Fig. 5.19). Measure the angles subtended by arc AKB and arc CLD at the centre O. If the angle at the centre is less than 180°, it is a minor arc. If the angle at the centre is greater than 180°, it is a major arc. State whether arcs AKB and CLD are minor arcs or major arcs.

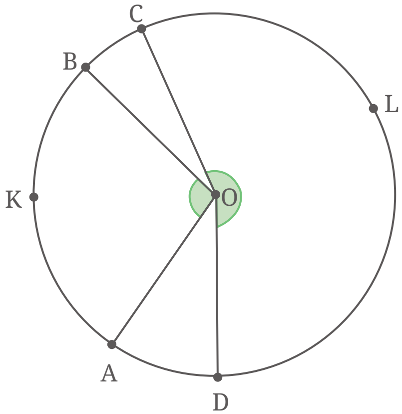


Fig. 5.19

5.7.1 Angle subtended by an arc at a point on the circle outside the arc

By 'angle subtended by arc AXB at a point on the circle outside the arc' we mean the angle ACB where C is any point on the circle but not on arc AXB. Remarkably, the measure of this angle does not depend upon which specific point C we pick, so long as it is on the circle and outside the arc!

Activity: Draw a circle and a chord AB. Fix an arc AKB formed by AB and a point K between A, B on the circle. Measure the angle subtended at the centre by arc AKB. Take three points P, Q, R on the circle outside arc AKB. Measure the angles subtended by arc AKB at points P, Q, R. What do you notice?

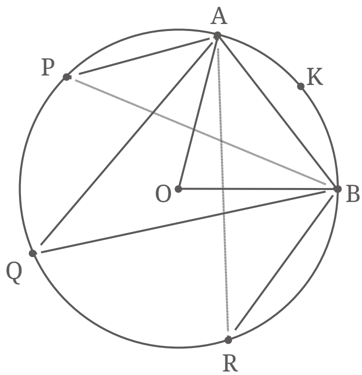


Fig. 5.20

Repeat this activity for a different arc AKB. Based on this activity, we can make a statement about the angles subtended by an arc of a circle at the centre and at a point of the circle outside the arc. We shall show that the following is true.

Theorem 9: *The angle subtended by an arc at the centre of the circle is double the angle subtended by the arc at any point on the circle outside the arc.*

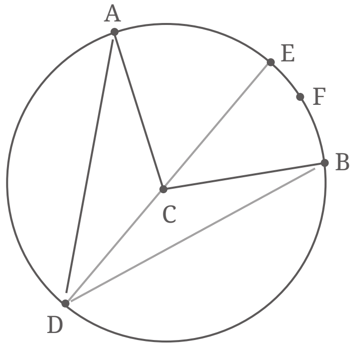


Fig. 5.21: Angle subtended by arc AFB

Given: AFB is an arc. ∠ACB is the angle subtended by arc AFB at centre C. D is a point on the circle outside arc AFB (see Fig. 5.21).

We will assume for now that D is such that DC when extended cuts the circle at some point E on arc AFB, as shown.

We will consider the other case later.

To show: ∠BCA = 2 ∠BDA.

Why is this true? Join D to C and extend DC to cut the circle at a point E on arc AFB. Now ΔDCB is isosceles with CB = CD. So, ∠CBD = ∠CDB.

∠BCE is the exterior angle of ΔBCD. By the exterior angle theorem,

∠BCE = ∠CBD + ∠CDB = 2 ∠BDC.

Similarly, ΔADC is an isosceles triangle with CA = CD.

So, ∠CAD = ∠CDA

∠ACE is the exterior angle of ΔADC. By the exterior angle theorem,

∠ACE = ∠CAD + ∠CDA = 2 ∠CDA.

Now ∠BCA = ∠BCE + ∠ECA, and ∠BDA = ∠BDE + ∠EDA.

Hence ∠BCA = 2 (∠BDC + ∠CDA) = 2 ∠BDA.

So, ∠BCA = 2 ∠BDA.

The explanation we have given does not work if the situation is as shown in Fig. 5.22, where the extension of DC meets the circle at a point E outside the arc AFB. We must rework the explanation for this case.

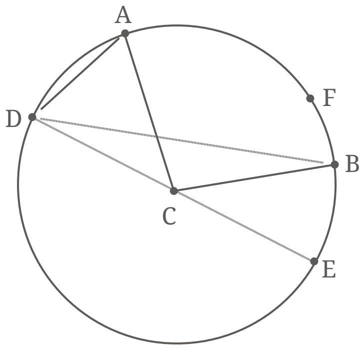


Fig 5.22: Angle subtended by arc AFB

Given: AFB is an arc. ∠ACB is the angle subtended by arc AFB at centre C. D is a point on the circle outside arc AFB such that when we extend DC it cuts the circle at some point E outside the arc AFB, as shown (Fig. 5.22).

We shall again make use of the properties of isosceles triangles.

∠ACE = ∠ADC + ∠CAD = 2 ∠ADC, since CA = CD (and so ∠ADC = ∠CAD).

Similarly, ∠BCE = ∠BDC + ∠CBD = 2 ∠BDC.

Also, ∠ACB = ∠ACE – ∠BCE, and ∠ADB = ∠ADC – ∠BDC.

Hence ∠ACB = 2 ∠ADB.

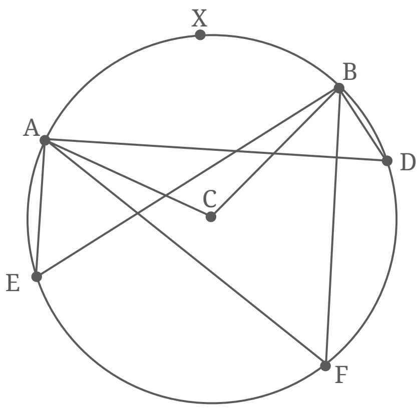


Fig. 5.23: Angles subtended by an arc are equal

Something very interesting comes out of this. This is what the activity before Theorem 9 suggested. Take an arc AB. Decide how you want to go from A to B along the arc. Take any point D on the circle, outside arc AB.

Then, no matter where D is, so long as it is outside the arc and on the circle, ∠ADB is the same!

Let us understand this better using Fig. 5.23. Consider the arc of the circle going from A to B via X. The points on the circle outside this arc are points you cross as you go from A to B via E. The angle subtended by AXB at the centre is the angle swept by CA as we move from CA to CB via X. Points E, F and D are all not on AXB. The theorem tells us that

∠AEB = ∠ADB = ∠AFB = $\frac{1}{2}$ ∠ACB.

Corollary: The angle subtended by a diameter at any point on the circle is 90°.

Corollary: A corollary is a fact that follows immediately from an already proved result.

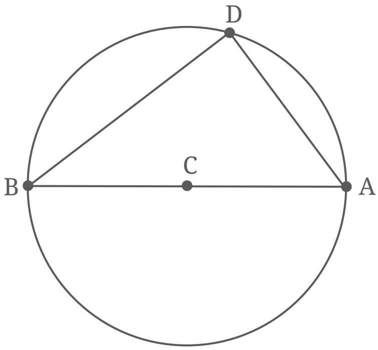


Fig. 5.24

Why is this true? Let AB be a diameter (Fig. 5.24). We must show that ∠ADB is 90°. The arc from A to B we take is the arc not containing D. What is the angle subtended by that arc at C? We have to move from A along that arc till we reach B. So, the angle subtended is ∠ACB, a straight angle, 180°. Hence, $\angle ADB = \frac{1}{2}\angle ACB = 90°$.

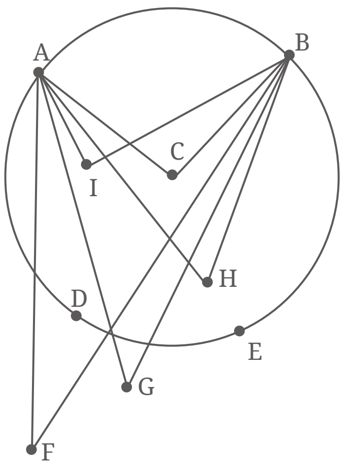


Fig. 5.25: Points and the chord

This fact that the angles in the same arc segment are all equal is a beautiful fact that distinguishes the circle from all other shapes.

In Fig. 5.25, the angles that arc AB subtends at the points F, G outside the circle are different. For the points inside the circle, ∠AIB, ∠ACB and ∠AHB are all different. But for all points P on arc AB, the subtended angles are the same. Thus, for points D, E, ∠ADB = ∠AEB.

Exercise Set 5.6

1. In a circle with centre O, the central angle AOB is 60°. If the radius of the circle is 12 cm, what is the length of the chord AB?

2. Let A and B be two points on a circle with centre O.
   - (i) Are there points X, Y on the circle, on the same side of AB, such that ∠AXB is different from ∠AYB?
   - (ii) Is it true that if ∠AXB = ∠AYB, then X and Y lie on the same side of the circle?
   - (iii) If ∠AXB = ∠AYB, and X and Y do not lie on the circle, does the circle through A, B and X also pass through Y?
3. Find x in Fig. 5.26.

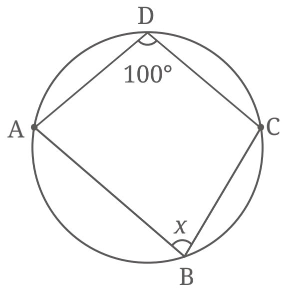


Fig. 5.26

5.8 Concyclicity of Points

Now we address the question about when 4 points lie on the same circle. And we will let you explore the question for 5 points, 6 points and so on! We call points that are on the same circle **concyclic**.

Theorem 10: *If a line segment* AB *joining two points* A, B *subtends equal angles at two other points* C, D *that lie on the same side of* AB, *then the four points lie on a circle.*

As always we draw the picture first.

Given: Consider segment AB. Points C and D lie on the same side of AB; points C and D are not on the line AB; and ∠ACB = ∠ADB.

To show: A, B, C, D lie on the same circle.

Why is this true? The points A, B, C are noncollinear points. So, using Theorem 1, there is a circle passing through A, B, C.

Let us show that D also lies on that circle. For this, let us draw the circle through A, B and C.

Suppose D is not on the circle; then it is either inside or outside the circle. Join AD.

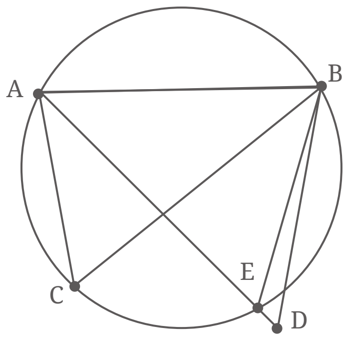


Fig. 5.27A

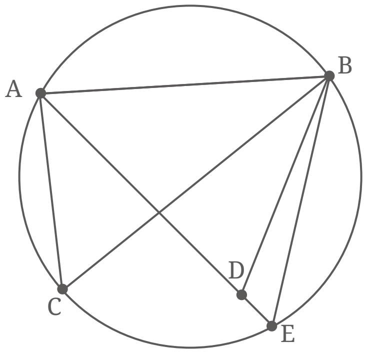


Fig. 5.27B

If D is outside the circle, then AD intersects the circle at E; see Fig. 5.27A. If D is inside the circle, extend AD to meet the circle at E; see Fig. 5.27B. Now C, E are on the same segment of the arc formed by chord AB. So, ∠ACB = ∠AEB.

If D were outside the circle, then ∠AEB is an exterior angle of the ∆BED, so ∠AEB > ∠ADB. Also, ∠AEB = ∠ACB (angles in the same segment of a circle), and ∠ACB = ∠ADB (given). This leads to ∠ACB being greater than itself, which is obviously not possible.

If D were inside as in Fig. 5.27B, then ∠ADB is an exterior angle of BED. In the same way that we argued above, we reach an impossible conclusion.

So, we must eliminate both of these options (D being outside the circle and D being inside the circle). So, D must be on the circle passing through A, B, C. In other words, A, B, C, D are concyclic.

When the vertices of a quadrilateral (which we also call a **4-gon**) are concyclic, the quadrilateral is called a **cyclic** quadrilateral.

We use this to show another nice property of circles!

Theorem 11: *The sum of two opposite angles of a cyclic quadrilateral is* 180°.

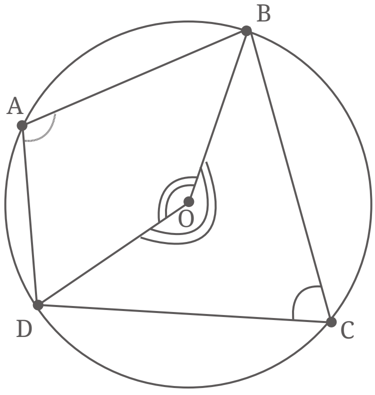


Fig. 5.28

Given: A, B, C, D are the vertices of a cyclic 4-gon (Fig. 5.28). This means that there is a circle passing through A, B, C, D. Let its centre be O.

To show: ∠BAD + ∠BCD = 180°.

How shall we show that this is true? Consider arc BCD; point A is on the circle and it lies outside the arc BCD. So ∠BAD is half the angle that arc BCD subtends at the centre O. Since we move from OB to OD along C, this is the reflex angle BOD (dotted). So, ∠BAD = $\frac{1}{2}$ (reflex angle BOD).

Similarly, ∠BCD is half the angle subtended by the arc BAD at O. This is the angle denoted by the double arc BOD. Since C is on arc BCD, we need to move from OB to OD via A. So, ∠BCD = $\frac{1}{2}$ (∠BOD).

So, ∠BAD + ∠BCD = $\frac{1}{2}$ (complete angle at the centre O)

A complete rotation at O is 360°.

Hence, ∠BAD + ∠BCD = $\frac{1}{2} \times (360°) = 180°$.

So, the opposite angles of a cyclic 4-gon add up to 180°.

The converse of this theorem also holds.

Exercise: A cyclic quadrilateral has angles measuring ∠A = 80°, ∠B = 110°, ∠C = 100°, and ∠D = 70°. Can such a quadrilateral be drawn? Explain why or why not.

Theorem 12: *If two opposite angles of a quadrilateral add up to 180°, then the vertices of the quadrilateral lie on a circle, i.e., they are concyclic.*

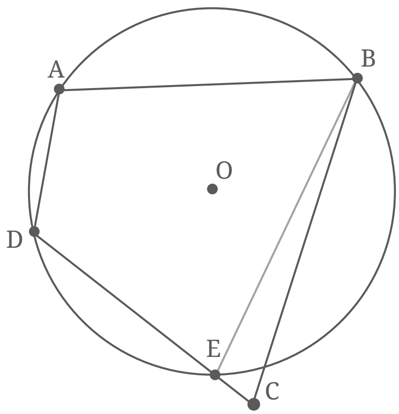


Fig. 5.29

Given: ABCD is a 4-gon (Fig. 5.29).

∠BAD + ∠BCD = 180°.

∠ABC + ∠CDA =180°.

To show that: ABCD is cyclic.

Why is this true? Suppose ABCD is not cyclic. Since A, D, B are not collinear, there is a circle passing through the three points. If that circle does not pass through C, there are two cases: C is outside or C is inside the circle.

We deal only with the first case and leave the other to the reader.

Let E be the point where CD meets the circle (see Fig. 5.29). Then ABED is a cyclic 4-gon. So from Theorem 11, ∠BAD + ∠BED = 180°.

We are also given ∠BAD + ∠DCB = 180°.

So ∠BCD = ∠BED. But this is not possible as ∠BED is the exterior angle of BEC and hence greater than ∠BCE, i.e., greater than ∠BCD.

So, points A, B, C, D all lie on a circle.

The circle is a beautiful and tantalising shape. After mastering these basic properties, you will get a feel for the symmetries of a circle and the properties of chords, the angles they subtend, the distances of chords from the centre, and angles subtended by arcs at the centre.

Next year you will study many more interesting and beautiful such results about circles. Stay tuned—the best is yet to come!

End-of-Chapter Exercises

1. In a circle, a chord is 5 cm away from the centre. If the radius of the circle is 13 cm, what is the length of the chord?
2. An arc of a circle subtends an angle of 70° at the centre. What is the measure of the angle subtended by the arc at a point on the circle?
3. The diameter of a circle is 26 cm. A chord of length 24 cm is drawn in the circle. Find the distance from the centre of the circle to the chord.
4. A circle has a radius of 15 cm. A chord is drawn. The distance from the centre of the circle to the chord is 9 cm. What is the length of the chord?
5. Prove that the perpendicular bisector of a chord passes through the centre of the circle.
6. The diameter of a circle is AB. Point C is on the circumference. What is the measure of the $\angle$ACB? Explain your reasoning.
7. ABCD is a cyclic quadrilateral inscribed in a circle. If $\angle$A measures 75°, what is the measure of $\angle$C? If $\angle$B measures 110°, what is the measure of $\angle$D?
8. Quadrilateral PQRS is inscribed in a circle. If $\angle P = (2x + 10)°$ and $\angle R = (3x - 20)°$, find the value of x and the measures of $\angle$P and $\angle$R.
9. The distance of a chord of length 16 cm from the centre of a circle is 6 cm. Find the radius of the circle.
10. A cyclic quadrilateral has sides 5, 5, 12, 12 units. Find its area.

*11. Consider a cyclic quadrilateral. Without drawing its circumcircle, how can we find out whether the centre of the circumcircle lies inside the quadrilateral or outside? What is the best way of finding out?

*12. When two chords intersect, each of them is divided into two line segments. Show that if the intersecting chords are of equal length, then the line segments of one chord are equal to the corresponding line segments of the other chord.

*13. Draw a circle in which a chord of 6 cm length stands at a distance of 3 cm from the centre.

(**Hint:** Is it a circumcircle of a suitable triangle?)

*14. Show that rectangle is the only parallelogram that can be inscribed in a circle.

*15. Show that if a rectangle is inscribed in a circle, then the point of intersection of its diagonals must lie at the centre of the circle.

*16. Consider all chords of a circle of a fixed length. What is the shape formed by the midpoints of all these chords?

*17. In a circle with centre O, chords AB and AC are congruent. Explain why this statement is true: "The centre of the circle lies on the angle bisector of ∠BAC".

18. Two parallel chords of lengths 10 cm and 24 cm are on the same side of the centre of a circle. The distance between the chords is 7 cm. Find the radius of the circle.

*19. A regular hexagon is inscribed in a circle of radius r. Find the length of the sides of the hexagon and the distance of each side from the centre of the circle.

20. A quadrilateral MNOP is inscribed in a circle. If MN is a diameter, what can you say about ∠MOP and ∠MNP? Explain your reasoning.

21. Let ABCD be a cyclic quadrilateral. Explain why the exterior angle at any vertex is equal to the interior opposite angle (e.g., ∠CDE = ∠ABC, where E is a point on the extension of side CD).

*22. "There is no chord of a circle that is longer than its diameter." How do you justify this statement?

*23. Let A be any point within a given circle with centre O. Show that the shortest chord of the circle that passes through point A is the one that is perpendicular to OA.

24. How would you use the following figure to justify the statement that the angle in a semicircle is 90°?

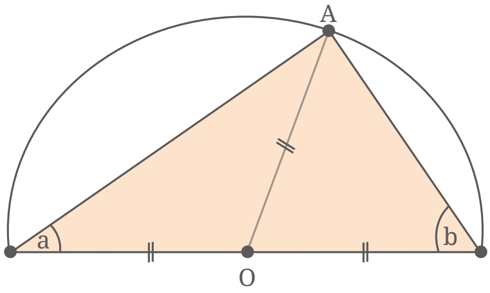


Fig. 5.30

*25. In a circle, two chords CC' and DD' are drawn perpendicular to a diameter AB. Prove that the segment MM' joining the midpoints of the chords CD and C' D' is perpendicular to AB.

*26. How would you use the following figure to justify the statement that the sum of the opposite angles of a cyclic quadrilateral is 180°?

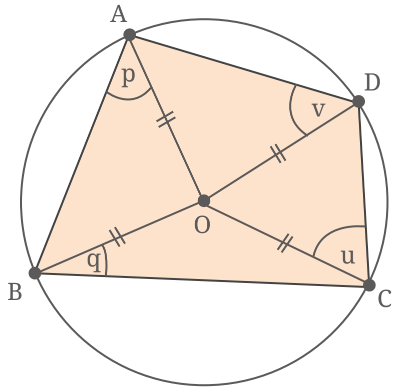


Fig. 5.31

Chapter Summary

- A **circle** is the set of all points in a plane that lie at a given distance (the **radius**) from a fixed point called its **centre**.
- A circle has **reflection symmetry** across any diameter.

- A circle has **rotational symmetry** about its centre, through any angle.
- Infinitely many circles can be drawn through two given points. The centres of these circles lie on the perpendicular bisector of the line segment joining the two given points.
- Given any three points not on a straight line, a **unique circle** can be drawn through them; it is called the **circumcircle** of the triangle whose vertices are the three given points. The centre of this circle is called the **circumcentre** of the three points, lies at the intersection of the perpendicular bisectors of the line segments joining the points.
- Equal chords subtend equal angles at the centre of the circle. Conversely, if two chords subtend equal angles at the centre, the chords are equal in length.
- A line drawn from the centre to the midpoint of a chord is perpendicular to that chord. Conversely, a perpendicular from the centre to a chord bisects it.
- Chords of equal length lie at equal distance from the centre. Conversely, chords that lie at the same distance from the centre are equal in length.
- Given two unequal chords, the longer chord is closer to the centre.
- The angle subtended by an arc at the centre of a circle is twice the angle it subtends at any point on the remaining part of the circle.
- The angle subtended by a diameter at any point on the circle is 90°.
- If a line segment between two points subtends equal angles at two other points on the same side of the segment, all four points lie on a single circle (i.e., the points are **concyclic**).
- A quadrilateral inscribed in a circle is called **cyclic**. In a cyclic quadrilateral, the sum of opposite pairs of angles is 180°. Conversely, if two opposite angles of a quadrilateral add up to 180°, then it is a cyclic quadrilateral.

6 Measuring Space: Perimeter and Area

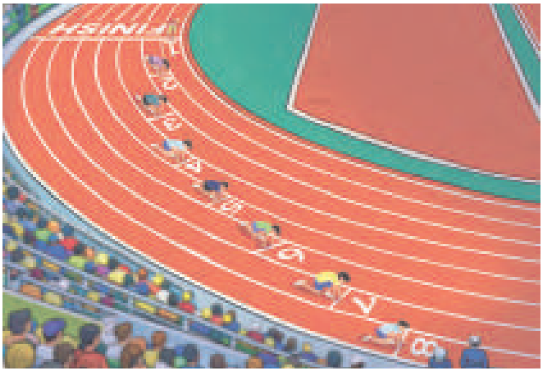


Fig. 6.1: Athletes at the start of 4 × 100 m relay race

In Fig. 6.1, you see athletes assembled at the start of a 4 × 100 m relay race. The tracks are laid out, and the athletes are all set to go racing down the tracks. Do you notice that the athletes are not at the same starting line?

Those in the outer lanes seem to be starting ahead of those in the inner lanes while the finish line is the same for all of them. What could be the reason for this? The distance between the starting points of adjacent lanes is called the 'stagger'. Notice that the stagger continues all the way to the outermost lane. Do you think the stagger gives anyone (those in the outer lanes or in the inner lanes) an unfair advantage? Why or why not? On what basis can the organisers work out the length of the stagger between lanes?

Think and Reflect

In my school, the playground is too small to have a 400 m track, so the school constructed a 200 m track instead. Does this mean that we need a smaller stagger for the race tracks in my school (i.e., smaller than the stagger used in the Olympics), for the same 4 × 100 m relay race?

To answer the question about the lane staggers required on a 4 × 100 m athletics track, we need to know how to find the length around a circle.

6.1 Perimeter of a Shape

Given any shape, its **perimeter** is the total length around its border. Imagine a tiny insect going for a walk around its border, never turning around, till it returns to its starting point. The perimeter of the shape is the total distance it travels.

So, a square with side a units has perimeter $4a$ units. An equilateral triangle with side a units has perimeter $3a$ units.

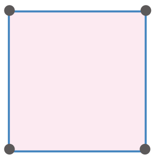


Fig. 6.2A

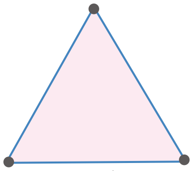


Fig. 6.2B

The perimeter of a rectangle with length a units and width b units is $2(a + b)$ units. Note that the formula for the perimeter of a square is a 'special case' of the formula for the perimeter of a rectangle with $a = b$.

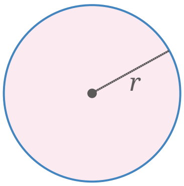


Fig. 6.3

Here we see a circle with radius r units. What is its perimeter? How do we find out?

Think and Reflect

What is the connection between this question and the one about the 400 m athletics track?

To answer the question about the perimeter of the circle, we must go step by step.

What happens to the perimeter of a square if we double its side? It doubles too. The ratio of perimeter to the side is 4:1; this is so for all squares. As the side of the square gets larger (or smaller), the ratio of perimeter to side stays fixed at 4:1.

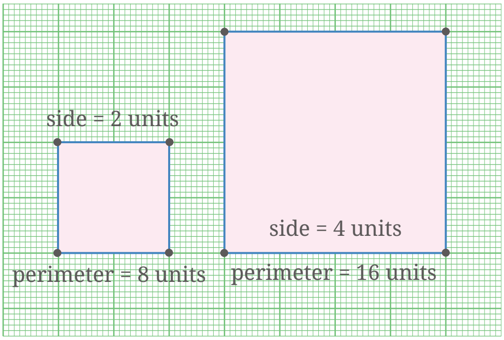


Fig. 6.4: Ratio of perimeter to side is 4:1

For equilateral triangles, the ratio of perimeter to the side is 3:1. As the side of the equilateral triangle gets larger (or smaller), the ratio of perimeter to side stays fixed at 3:1.

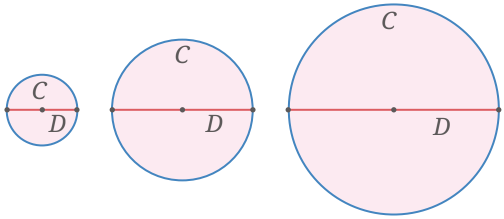


Fig. 6.5

What about a circle? What is its perimeter (usually called the **circumference**) in terms of its diameter?

Is the ratio of circumference (C) to diameter (D) the same for circles of all sizes (Fig. 6.5)? What do you think?

6.2 Perimeter of a Circle — The *C*/*D* Ratio

In ancient days people realised that the ratio of the circumference to the diameter of the circle does not change if we change the size of the circle.

Let's call this ratio the 'C/D ratio' of the circle.

What is the value of the C/D ratio?

How would you estimate this ratio?

HOME MEASUREMENT

You can do a simple measurement at home to estimate the C/D ratio. Take a cotton reel with thin thread around it. Measure the diameter D of the reel as accurately as possible. Unwrap and then tightly wrap the thread around the reel 20 times. Unwrap it again; measure its length L, and calculate $\frac{L}{20\,D}$. This is the ratio we want. For accuracy, the thread should be very thin. Please do the experiment! Do you get a ratio between 3 and 4? Between 3.1 and 3.2?

It is also possible to estimate the C/D ratio using pure geometry, i.e., without any measurements at all! Can you imagine how?

C/*D*'s Adventurous Journey: From Ancient Approximations to the Exact Formula of Mādhava

Mathematicians have been fascinated by circles, and the C/D ratio, since ancient times and across geographical regions. This constant, which we now call π (we say 'pi' as in 'pie' and not as in 'pizza'), represents a bridge between many different areas of mathematics, and connects the straight-edged world of polygons with the infinite curves of nature. While early civilisations relied on practical, 'good-enough' values for construction and trade, the pursuit of this ratio eventually became an important pastime of mathematicians that helped to benchmark the sophistication of mathematical theories.

In Mesopotamia (c. 1900 BCE), mathematicians moved beyond the crude integer value of 3. They realised that a circle had a perimeter that was slightly larger than the hexagon inscribed within it. They concluded that π should therefore be larger than 3, and they set the value of π to be $3 + \frac{1}{8} = 3.125$.

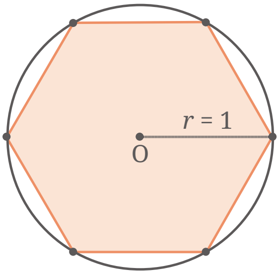


Fig. 6.6 The Mesopotamian Hexagon-to-Circle comparison. Can you see why this shows that π > 3?

By **250 BCE, Archimedes of Syracuse** brought a new level of rigor to the problem. He 'trapped' the value of π between the perimeters of inscribed and circumscribed polygons. By using both an inscribed and circumscribed hexagon, Archimedes showed that π is between 3 and $2\sqrt{3} \approx 3.46$. Working his way up to 96-sided polygons, Archimedes found that $3\frac{10}{71} < \pi < 3\frac{1}{7}$.

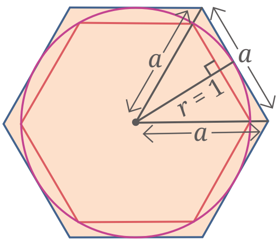


*Fig. 6.7: Archimedes' method utilising inscribed and circumscribed polygons. Can you see why this diagram of an inscribed and circumscribed hexagon tells us that π is between 3 and $2\sqrt{3}$? (**Hint:** Use the Baudhāyana–Pythagoras Theorem.)*

In about 150 CE, Ptolemy of Alexandria refined Archimedes computations for use in his astronomical tables, giving the ratio $\frac{377}{120} \approx 3.14167$ for π.

Shortly thereafter, in China, a 'circle-cutting method' of **Liu Hui** (263 CE) laid the groundwork for two breakthroughs due to **Zu Chongzhi** (480 CE). Zu pushed the polygon approximation method to 24,576 sides! He used this to discover the Yuelü (Approximate Ratio) of $\frac{22}{7} \approx 3.1428$ for π, and also the *Miü* (Close Ratio) of $\frac{355}{113} \approx 3.1415929$ for π. The rational fraction $\frac{355}{113}$ is so close to π that it remained the most accurate value for π in the world for over 800 years; it is now known that no single rational fraction with denominator less than 15,000 can be as close to π as this fraction!

In 499 CE, Āryabhaṭa provided a value of $\frac{62832}{20000} = 3.1416$ for π. Crucially, he described this value as *asanna*, i.e., 'approaching' or 'approximate'— a profound insight suggesting that the ratio could not be given exactly as one simple fraction. Meanwhile, **Brahmagupta** (628 CE) suggested the use of the value $\sqrt{10} \approx 3.1622$ for π; while slightly less accurate, he chose it for its mathematical elegance and the ease with which it could be manipulated in equations, a preference that saw $\sqrt{10}$ become the dominant approximation in the Arab world and medieval Europe for centuries after.

Over the ensuing years, many great mathematicians studied π, including its approximations; but no approximation was as accurate as Zu's $\frac{355}{113}$— or as algebraically simple as $\frac{22}{7}$, 3.1416, and $\sqrt{10}$ —until the work of Mādhava of Sangamagrāma, who changed the game forever. Mādhava realised that π was not just a number to be approximated by fractions and other finite algebraic expressions, but a limit to be reached. Mādhava thereby discovered the first exact formula for π:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Is that not a beautiful formula? Mādhava's formula—given in the form of an 'infinite series'—was a tectonic shift for mathematics. By moving from the geometric cutting of circles to the analytical summing of numbers, Mādhava birthed the area of mathematics known as calculus. His infinite series enabled him to calculate π to 11 decimal places (3.14159265358), proving that the relationship between a circle's circumference and its diameter was a window into an entirely new area of mathematics.

Better and better infinite series were developed over time that gave more and more digits of π even more quickly. The key breakthroughs on the problem after Mādhava were due to Nīlakaṇṭha (c. 1500), Machin (1706), Ramanujan (1914), and the Chudnovsky brothers (1988) who skilfully extended the ideas of Ramanujan. Today, using their algorithms, we now know π to neels (100s of trillions) of digits! We will learn more about the area of calculus in later grades.

In 1706, the Welsh mathematician William Jones used the Greek symbol π to denote the *C*/*D* ratio, because π is the first letter of the Greek word *perimetros* for perimeter. The symbol was made popular by the Swiss German mathematician Leonhard Euler (pronounced 'oiler'). We continue to use the same symbol today!

6.3 π Is Irrational

The digits of π go on forever, with no visible pattern. You already know that fractions give rise to decimal expansions with a rhythmic pattern, e.g.,

$$\frac{1}{3} = 0.33333..., \frac{1}{11} = 0.09090909..., \frac{1}{7} = 0.142857\ 142857...$$

But for π, there is no such pattern! It turns out that π cannot be written as a ratio of two integers. Such numbers are called *irrational*. Remember: A rational number is a number that can be written as a fraction $\frac{a}{b}$ where *a*, *b* are integers with *b* not equal to 0, e.g., numbers such as $1\frac{2}{3}$, $\frac{-7}{11}$ and 1.4.

In Grade 8, you learned that $\sqrt{2}$ is an irrational number. From their writings, it seems that Āryabhaṭa and Zu Chongzhi regarded π as irrational. Centuries later (1761), the mathematician Lambert showed that this is so; π is irrational. But to prove this requires more advanced mathematics, which we will learn much later.

Since π is irrational, there is no 'best fraction' for π. If there is any fraction that is close to π, we can find another fraction that is closer.

The well-known fraction $\frac{22}{7}$ is good enough for most practical uses. But it is important to note that π is not equal to $\frac{22}{7}$ (they are 'close but not equal'). So we write $\pi \approx \frac{22}{7}$ and also $\pi \neq \frac{22}{7}$. In the same way, we can write $\sqrt{2} \approx 1.414$ and also $\sqrt{2} \neq 1.414$.

As noted earlier, a much better approximation for π is $\frac{355}{113}$.

Fun Fact

Here's a fun way to remember the first few digits of π:

How I wish I could recollect pi.

Count the number of letters in each word (3, 1, 4, 1, 5, 9, 2) and you get the digits of π! So, $\pi \approx 3.141592\ldots$

Since $\pi \approx 3.14$ and also $\pi \approx \frac{22}{7}$, March 14 (written '3-14' in North America) is celebrated each year as **Pi Day**, and 22 July (written '22-7' in

India) is celebrated each year as **Pi Approximation Day**.

6.4 Length of an Arc of a Circle

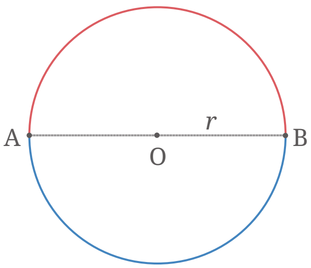


Fig. 6.8: Two semicircles making a full circle

The circumference of a circle of diameter d is π *d*. Since the diameter is twice the radius, we can also write the circumference of a circle as $2\pi r$, where r is the radius.

What will be the length of a **semicircle** with the same radius r (see Fig. 6.8)?

If we reflect the circle in the diameter AB, the red semicircle and the blue semicircle exchange places. This means that they have equal length, which is $2\pi r \div 2 = \pi r$.

Instead of reflecting the figure in the diameter AB, we can also imagine rotating the whole figure around the centre O through an angle of 180°, i.e., through a half-turn. The effect is the same, and we get the same formula for the length of the semicircular arc.

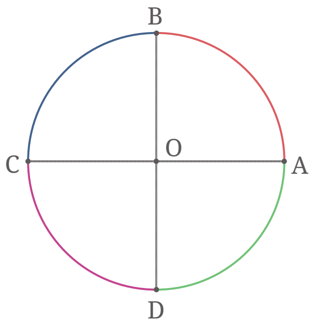


Fig. 6.9: Four quarter circles make a full circle

What will be the length of a **quarter circle** with the same radius (Fig. 6.9)?

Imagine rotating the entire figure through 90°, clockwise or anticlockwise. Each quarter circle moves and covers another quarter circle. So each quarter circle has the same arc length, and this is equal to $2\pi r \div 4 = \frac{\pi r}{2}$.

Note the following:

- The formula for the length of a semicircle may also be written as $2\pi r \times \frac{180^\circ}{360^\circ}$.

- The formula for the length of a quarter circle may also be written as $2\pi r \times \frac{90^\circ}{360^\circ}$.

From these, we can guess the formula for the length of an arc of a circle in terms of the angle it makes (i.e., the angle it 'subtends') at the centre of the circle.

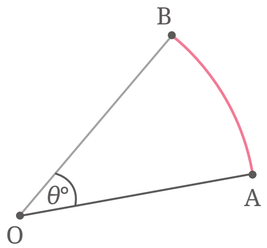


Fig. 6.10: Length of an arc of a circle

If the arc is AB, and it subtends an angle $\theta°$ at the centre O of the circle, the length of the arc is $2\pi r \times \frac{\theta^\circ}{360^\circ}$.

A Closer Look at a 400 m Athletics Track

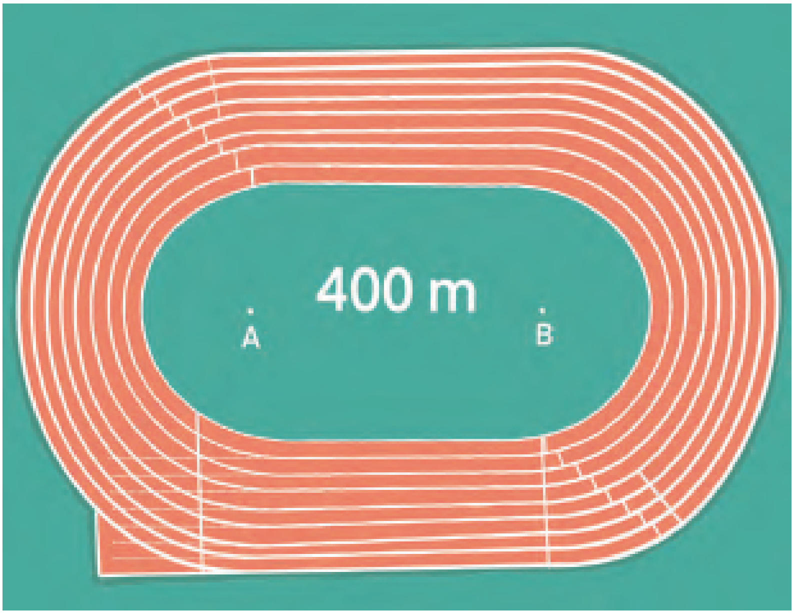


Fig. 6.11: Schematic diagram of a 400 m athletics track

Fig. 6.11 depicts a 400 m athletics track. You can see two straight sections of length 84.39 m each, and two curved portions, which are semicircles with a common centre (points A and B); the innermost semicircle on each side has radius 36.5 m. The width of each lane is 1.22 m.

Let an athlete make one complete circuit of the track. What is the total distance she runs?

- Let us assume that she runs at a distance of 0.3 m from the inner border.
- She runs two straight sections of length 84.39 m each, a total of 168.78 m.
- She also runs two semicircles of radius (36.5 + 0.3) m, i.e., 36.8 m.
- The two semicircles together make up a complete circle.
- Its circumference is $2 \times \pi \times 36.8 = 2 \times 3.1416 \times 36.8 = 231.22$ m.
- The total distance run by the athlete is therefore 168.78 + 231.22 = 400 m.

Now consider two runners, one in the innermost lane, the other in the second lane. On the straight stretch the two runners run the same distance. But on the curved portions the runner in the second lane runs a greater distance, because her semicircle has a larger radius. It is to compensate for this that staggers are needed.

Think and Reflect

What is the difference in radius between the first and second lanes? Use the Fig. 6.11 to find the stagger needed by the runner in the second lane. Will an equal stagger be needed between the third and second lanes?

6.5 Problems, Puzzles, and Paradoxes on Perimeter

We close this section by looking at a sprinkling of interesting problems around the theme of perimeter.

Example 1: Two circles of equal radius are located such that each circle passes through the centre of the other circle (Fig. 6.12).

Given that the radius of each circle is r units, find the perimeter of the shape formed by the two circles in terms of r units. (Ignore the dotted portions that lie within the circles.)

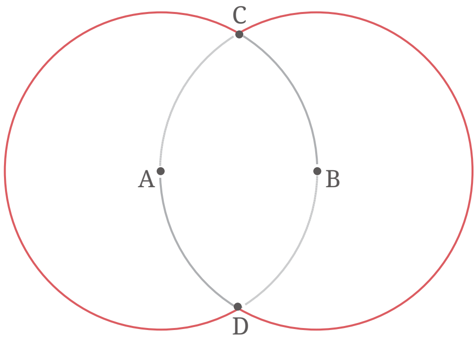


Fig. 6.12

Here, we have two congruent circles centered at A and B. Each one passes through the centre of the other one. The circles intersect at C and D. We need to find the total length of the two red arcs shown here, in terms of the radius r unit.

Consider the triangle with vertices A, B, C. Since AB = r, AC = r, BC = r, the triangle is equilateral, hence ∠CAB = 60° = ∠CBA.

Similarly ∠BAD = 60° = ∠ABD. It follows that ∠CAD = 120° = ∠CBD.

Hence, each dotted arc is $\frac{1}{3}$ of the circumference of the circle on which it lies. Hence, the total length of the two red arcs is $2 \times \frac{2}{3} \times 2\pi r = \frac{8}{3}\pi r$.

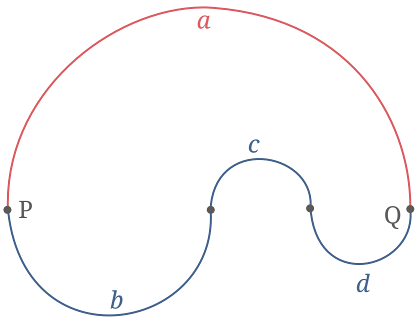


Fig. 6.13

Example 2: In Fig. 6.13, we see points P and Q and two paths connecting them. The first path is made up of the semicircle a. The other path is made up of three semicircles (b, c and d). Which path is longer? Choose one: (i) Path a is longer. (ii) Path $b + c + d$ is longer. (iii) The two paths have equal length. (Try to answer this before reading on.)

Let us solve the problem using algebra. Let the radii of the semicircles a, b, c, d be denoted by a', b', c' and d'. Then the length of semicircle a is $\frac{1}{2}$ (2π) a' = πa'. Similarly, the lengths of the semicircles b, c, and d are πb', πc' and πd'. So, the length of the second path is π($b' + c' + d'$), and that of the first path is πa'.

Which is bigger, $b' + c' + d'$ or a'? Do you see that neither one is bigger? They are equal! Because, the length of PQ is $2a'$ and it is also equal to $2b' + 2c' + 2d'$. It follows that $a' = b' + c' + d'$. Hence, the two paths have equal length!

Exercise Set 6.1

Unless stated otherwise, use the approximation $\frac{22}{7}$ for π.

1. The perimeter of a circle is 44 cm. What is its radius?
2. Calculate, correct to 3 significant figures, the circumference of a circle with: (i) radius 7 cm (ii) radius 10 cm (iii) radius 12 cm.
3. Calculate the length of the arc of a circle if: (i) the radius is 3.5 cm and the angle at the centre is 60°, and (ii) the radius is 6.3 m and the angle at the centre is 120°.
4. Find the perimeter of a sector (i.e., the curved portion as well as the two straight portions) of a circle of radius 14 cm and sector angle 75°.
5. Find the perimeters of the following shapes (taking the arcs to be quarter or half or three-quarters of a circle, as appropriate) (Fig. 6.14i to 6.14ix):

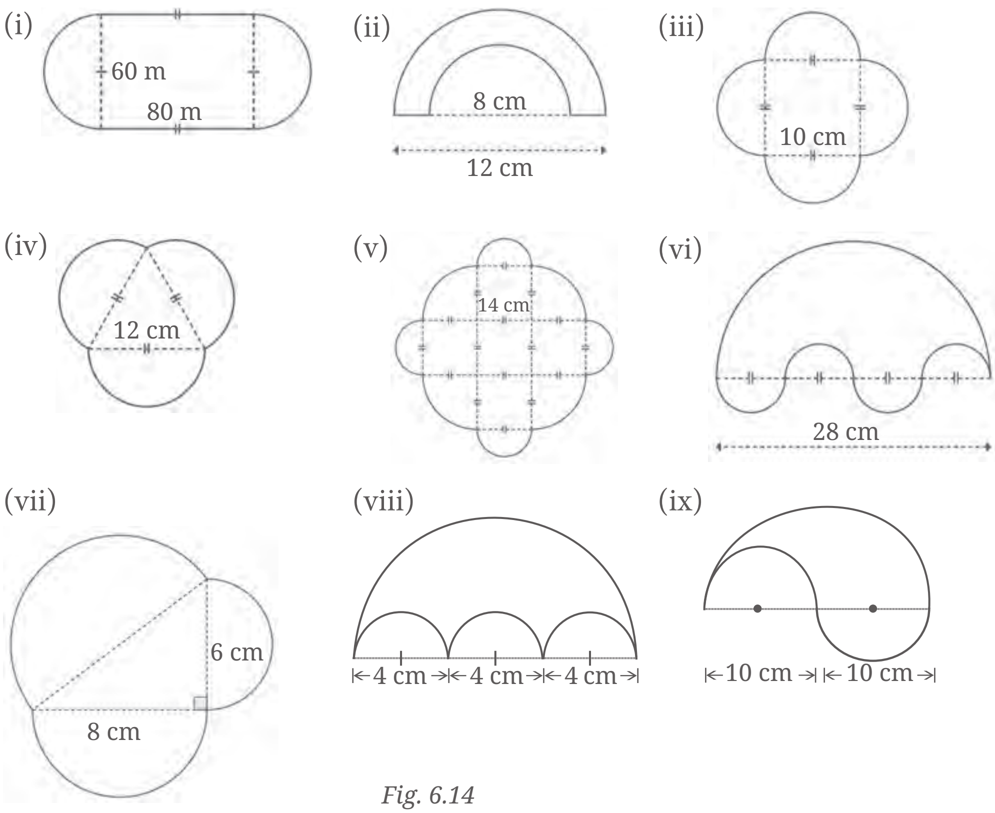


Fig. 6.14

6. If the diameter of a car tyre is 56 cm, then: (i) How far does the car need to travel for the tyre to complete one revolution? (ii) How many revolutions does the tyre make if the car travels 10 km?

7. Find the total perimeter of all the petals in each of the given flowers.

(i)

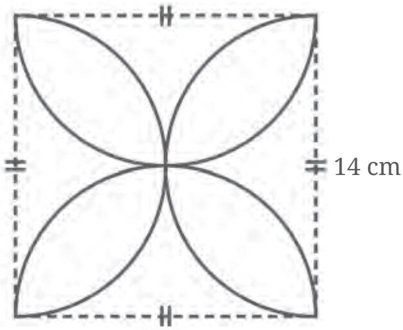


Fig. 6.15A: The centres of the arcs are the midpoints of the sides of the square

(ii)

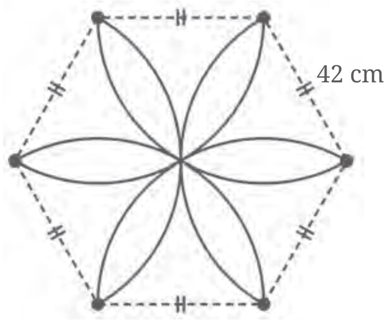


Fig. 6.15B: The centres of the arcs are the vertices of the hexagon

8. The ratio of the perimeters of two circles is 5:4. What is the ratio of their radii?

6.6 Area of a Rectangle

From perimeter, we move to area, i.e., to the ‘amount of space’ occupied by a two-dimensional region in a plane.

Measurement is always relative to something we call a unit. For area, the unit is a 1 × 1 square; its area is taken to be 1 unit^2 (also written as 1 sq. unit). You know from Grade 8 that the area of a rectangle with sides *a* units and *b* units is *ab* sq. units (Fig. 6.16).

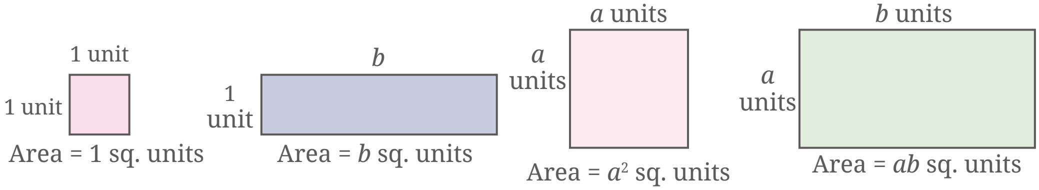


Fig. 6.16: Area of square and rectangle

6.7 Area of a Parallelogram

You also know from Grade 8 the formula for the area of a parallelogram.

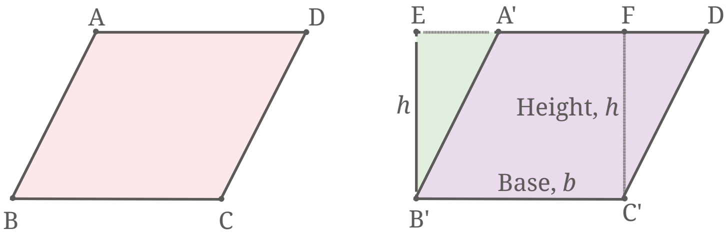


Fig. 6.17: Area of a parallelogram. Here, A'B'C'D' is a copy of ABCD.

Fig. 6.17 shows how to transform a parallelogram ABCD into a rectangle EB'C'F (at right; here, parallelogram A'B'C'D' is a copy of ABCD) with the same base b and same height h. Though the two shapes are different, their areas are the same. So, the area of the parallelogram is equal to base × height = bh.

Think and Reflect

What happens if the parallelogram is 'thin' (Fig. 6.18) and the foot of the perpendicular from C to AD does not lie on side AD? The construction then does not seem to work. How do we fix this 'gap'?

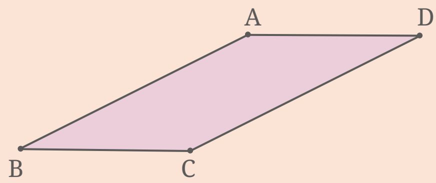


Fig. 6.18

Here is a hint on how the problem of the 'thin parallelogram' can be solved.

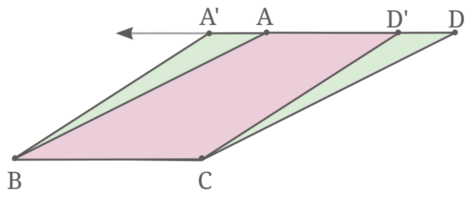


Fig. 6.19: The thin parallelogram

Select a point D' on DA, close to D, and a point A' on DA extended, with A'A = D'D (Fig. 6.19). Then A'BCD' is a parallelogram, and since ΔCDD' ≅ ΔBAA', its area is the same as that of ABCD. Now repeat this step as many times as needed.

Think and Reflect

The area of a rectangle can be found when we know the lengths of its sides. Is the same true for a parallelogram? That is, can we find the area of a parallelogram when we know the lengths of its sides? Why or why not?

(**Hint**: What happens to the area of a parallelogram if we decrease or increase the angle between the adjacent sides while keeping the lengths fixed?)

6.8 Area of a Triangle

Next, we work out a formula for the area of a triangle. (You have seen the formula in Grade 8, but we will go over the idea again, briefly.) One way to proceed is to first consider the case of a right-angled triangle and then other triangles. In both cases, we enclose the triangle in a rectangle; then we use the formula for area of a rectangle. (See Fig. 6.20A. Notice that FG = FJ + JG and so $b = b_1 + b_2$.) We find as a result that the area of a triangle with base b units and height h units is half the area of the rectangle with base b units and height h units. That is, the area is equal to $\frac{1}{2}bh$ sq. units.

You may wonder, like earlier, is there a gap in our argument? What would we do if angle EFG is obtuse and the triangle were shaped like triangle EFG in Fig. 6.20B? Please work out the answer to this question.

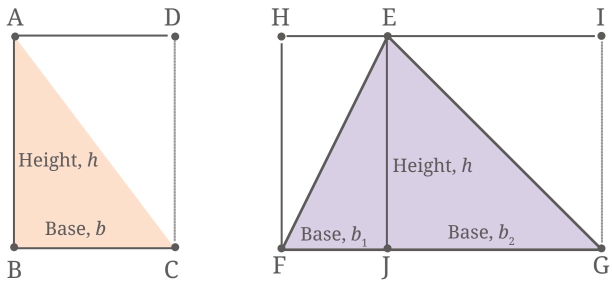


Fig. 6.20A: Area of a triangle

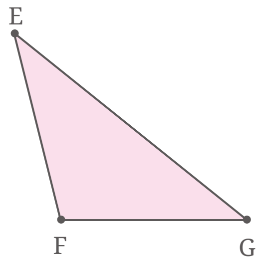


Fig. 6.20B

A nicer way of working out the formula for the area of a triangle is by seeing that two congruent copies of a triangle can be fitted together to make a parallelogram.

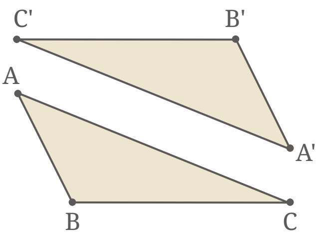


Fig. 6.21A

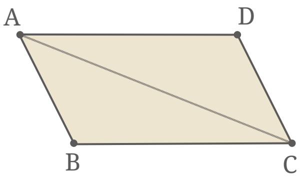


Fig. 6.21B

In Fig. 6.21A, triangles ABC and A'B'C' are congruent. They fit together as in Fig. 6.21B to make a parallelogram.

Do you see why the two triangles fit together to make a parallelogram? (If you study the angles in the figure (e.g., ∠B'C'A' and ∠BCA), you will see why this is so. Keep in mind the criterion by which we check whether two lines are parallel.)

We already know, area of a parallelogram: base × height = *bh*.

So, the formula for the area a triangle is $\frac{1}{2}$ (base × height) = $\frac{1}{2}$ *bh*.

A property of the median of a triangle. From the above formula we reach a simple but beautiful conclusion about any of the medians of a triangle.

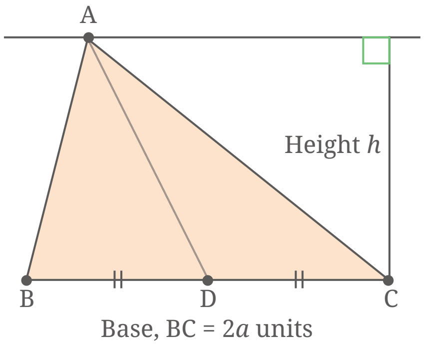


Fig. 6.22

A *median* is the line segment joining a vertex of the triangle to the midpoint of the opposite side. In Fig. 6.22, AD is a median of ΔABC.

Consider ΔABD and ΔACD. Their bases are equal (BD = DC), and they have equal height *h*. Using the formula for the area of a triangle, we see that they have equal area $\frac{ah}{2}$ sq. units!

We state this as a theorem.

Theorem: *A median of a triangle divides it into two triangles with equal area.*

Does this come as a surprise? It should! After all, ΔABD and ΔACD are (in general) differently shaped (i.e., not congruent to each other). But we have just proved that they have the same area!

Think and Reflect

Since ΔABD and ΔACD have equal area, you may wonder — Can we divide ΔABD using straight cuts into two or more pieces that we can then rearrange to exactly cover ΔACD? What do you think? Is it possible?

We shall spoil the surprise by revealing that it is possible. But we will not tell you the least number of pieces required. Try to find the answer!

Think and Reflect

Suppose we are given two polygons P and Q with equal area. Will it always be possible to divide one of them using straight cuts into two or more pieces and then rearrange the pieces to exactly cover the other polygon? Try this out for familiar shapes, e.g.,

1. A square and non-square rectangle with equal area,
2. Two triangles with different shapes but equal area,
3. A triangle and a square with equal area. Formulate a conjecture of your own about this.

Think of various rectangles with perimeter 40 units (the sides do not have to be integers).

1. How many such rectangles are there?
2. Among them, is there one whose area is the largest? What are its dimensions?
3. Among all these rectangles, is there one whose area is the smallest? What are its dimensions? Do either of these answers come as a surprise to you?

6.8.1 Heron's formula

You already know a formula for the area of a triangle (half base times height). Are there other formulas for the area of a triangle? There are several. For now we mention Heron's formula, discovered by the Greek mathematics and inventor, Heron. He taught at the Museum in Alexandria, a city in ancient Egypt located on the banks of the Nile.

The formula states that if ΔABC has side lengths BC = a, CA = b, and AB = c, then its area can be found as follows. First, we compute the semi-perimeter, s which is half the perimeter: $s = \frac{1}{2}(a+b+c)$. Then the area (ΔABC) is given by

$$\sqrt{s(s-a)(s-b)(s-c)}.$$

I am sure you will find the formula extremely surprising and strange looking! Let us test it against some known cases.

Example 3: An equilateral triangle with side a units.

All three sides have length a, so the semi-perimeter is $s = \frac{1}{2}(a + a + a) = \frac{3}{2}a$ units. Heron's formula now yields,

$$\text{area of triangle} = \sqrt{\frac{3}{2}a\left(\frac{1}{2}a\right)\left(\frac{1}{2}a\right)\left(\frac{1}{2}a\right)} = \frac{\sqrt{3}}{4}a^2 \text{ sq. units}$$

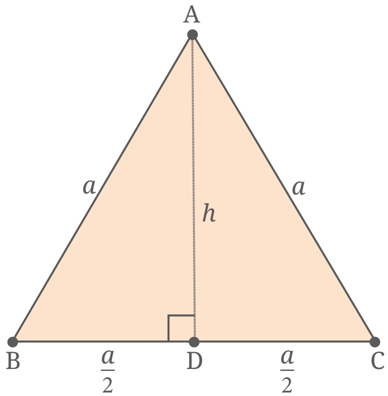


Fig. 6.23

Let us check this against the 'half base times height' formula. Let h units be the height of the triangle. Then, by the Baudhāyana–Pythagoras theorem,

$$a^2 - h^2 = \frac{a^2}{4}, \quad \therefore h^2 = \frac{3a^2}{4}, \quad \therefore h = \frac{\sqrt{3}}{2}a.$$

So the area is $\frac{1}{2} \times \frac{\sqrt{3}}{2}a \times a = \frac{\sqrt{3}}{4}a^2$ sq. units.

Same formula!

Note: Notice the symbol '∴'. This symbol is used regularly by mathematicians; it means and is read as 'therefore'.

Example 4: An isosceles triangle with equal sides a units and base $2b$ units.

The semi-perimeter is $s = \frac{1}{2}(a + a + 2b) = a + b$ units. Heron's formula now yields,

area of the triangle $= \sqrt{(a+b)(a+b-a)(a+b-a)(a+b-2b)} = b\sqrt{a^2 - b^2}$ sq. units.

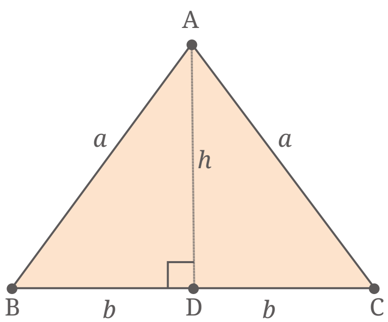


Fig. 6.24

Let us check this against the 'half base times height' formula. Let h units be the height of the triangle. Then, by the Baudhāyana–Pythagoras theorem,

$$a^2 - h^2 = b^2, \quad \therefore h^2 = a^2 - b^2, h = \sqrt{a^2 - b^2}.$$

So the area is $\frac{1}{2} \times \sqrt{a^2 - b^2} \times 2b = b\sqrt{a^2 - b^2}$ sq. units, the same as earlier.

Example 5: A triangle with sides 3 units, 4 units and 5 units.

The semi-perimeter is $s = \frac{1}{2}(3+4+5) = 6$ units. Heron's formula now yields,

$$\text{area of triangle } = \sqrt{6\times(6-3)\times(6-4)\times(6-5)} = \sqrt{6\times3\times2} = 6 \text{ sq. units.}$$

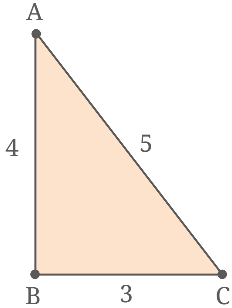


Fig. 6.25

Let us check this against the 'half base times height' formula. See Fig. 6.25; note that $3^2 + 4^2 = 5^2$. Therefore, by the converse of the Baudhāyana–Pythagoras hypotenuse theorem, ΔABC is right-angled at B. So,

base = 3 units, height = 4 units,

so the area is $\frac{1}{2}(\text{base}\times\text{height}) = \frac{1}{2}(3\times4) = 6$ sq. units, the same as earlier.

We get the correct result in each case.

You may wonder how Heron's formula is proved. There are many proofs known; one of them uses the Baudhāyana–Pythagoras theorem and repeated use of the 'difference-of-two-squares' formula $a^2 - b^2 = (a - b)(a + b)$. We will study some of these proofs in Grade 10.

There are two other such formulas for the area of a triangle. Both have a connection with circles. See Fig. 6.26.

Given any triangle ABC, there is exactly one circle that passes through its three vertices. This is called the **circumcircle** of ΔABC.

There is also exactly one circle that fits tightly in the triangle, touching its three sides. This is called the **incircle** of ΔABC.

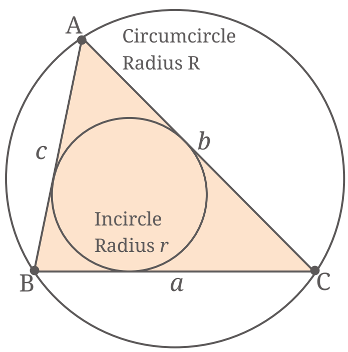


Fig. 6.26

Let the sides of ΔABC be a, b, and c. Let the radius of the circumcircle be R. Let the radius of the incircle be r. Then we have the following two formulas for the area:

$$\text{Area of } \Delta ABC = \frac{abc}{4R}$$

$$\text{Area of } \Delta ABC = \frac{r(a+b+c)}{2}$$

Both formulas have a beautiful symmetry about them! The proof of the second formula requires a result that you will study in Grade 10.

Brahmagupta's Formula for the Area of a Cyclic 4-gon

We have seen how to find the area of a triangle given the lengths of its three sides. The natural question then is: how can we find the area of a 4-gon given the lengths of its four sides?

Earlier we asked, can we find the area of a parallelogram if we know only the lengths of its sides? The answer is: No. In the same way we ask: can we find the area of a 4-gon if we only know the lengths of its sides? The figures below reveal the answer to this question.

The problem (Fig. 6.27) is about a 4-gon whose sides are known to be 3, 3, 3, 3 (it is a 'rhombus'). As you can see, the areas of the three figures are different. (We drew the figures using *GeoGebra* and found the areas using the 'Area' tool. Please try this exercise yourself, or by using four rods joined together at their ends.)

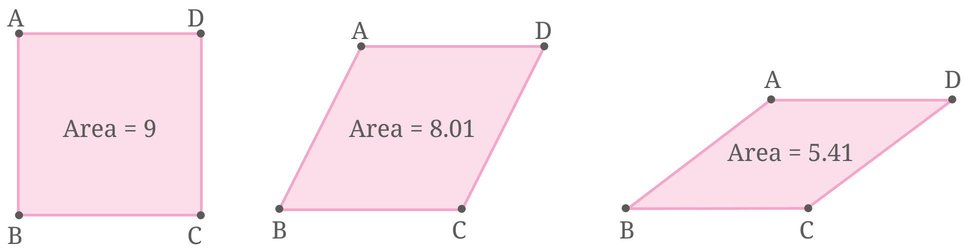


Fig. 6.27: The area of a 4-gon cannot be found only from the lengths of its sides

So we cannot find the area of a 4-gon only from the lengths of its four sides; we need more information. This could be one angle of the 4-gon, or the length of one diagonal, or the angle at which the diagonals cut each other, etc.

Or it could be a key geometric property of the figure.

One such property is that the 4-gon is a cyclic 4-gon. In 628 CE, Brahmagupta (598–668 CE) discovered a wonderful formula for the area of a **cyclic** 4-gon.

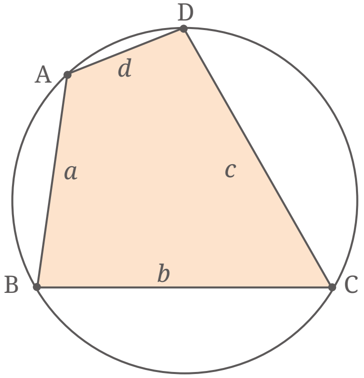


Fig. 6.28: A cyclic 4-gon

The formula states that if the sides of the cyclic 4-gon have lengths a, b, c, d, and the semi-perimeter s is $s = \frac{1}{2}(a+b+c+d)$, then:

$$\text{Area of 4-gon} = \sqrt{(s-a)(s-b)(s-c)(s-d)}.$$

It is an amazing and beautiful formula!

Does it not remind us of Heron's formula for the area of a triangle?

We can verify that the formula works out correctly in various special cases.

Example 6: Verify Brahmagupta's formula for the case of a rectangle.

All rectangles are cyclic, so Brahmagupta's formula should apply. Let the sides of the rectangle be a, b. Then $s = \frac{1}{2}(2a+2b) = a+b$, so the formula yields:

$$\text{Area} = \sqrt{(a+b-a)(a+b-b)(a+b-a)(a+b-b)} = \sqrt{ab \cdot ab} = ab,$$

which is correct. Please check out the formula against other special cases.

Example 7: Verify Brahmagupta's formula for the case of an isosceles trapezium.

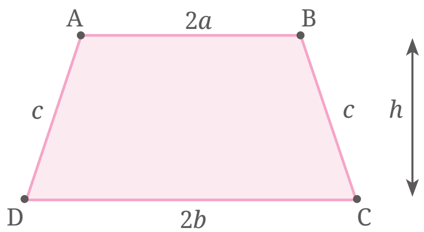


Fig. 6.29: Isosceles trapezium

All isosceles trapezia are cyclic, hence the formula should apply. The perimeter is $2a + 2b + 2c$, so $s = a + b + c$. Hence, the area as given by the formula should be

$$\sqrt{(s-2a)(s-2b)(s-c)(s-c)}.$$

Since $s - 2a = c + b - a$, $s - 2b = c + a - 2b$, $s - c = a + b$ the above formula simplifies to

$$(a+b)\sqrt{(c+b-a)(c+a-b)} = (a+b)\sqrt{c^2-(b-a)^2}.$$

If we imagine a perpendicular dropped from B to the base CD, its length (recall the Baudhāyana–Pythagoras theorem) is $\sqrt{c^2 - (b-a)^2}$. Hence, the area is $(a + b)h$. We get exactly the same formula if we recall that the area of an isosceles trapezium is half the sum of the parallel sides times the distance between the parallel sides.

Special Cases and Generalisation in Mathematics

The notion of a 'special case' occurs often in higher mathematics. A special case results from a general result whenever we apply some extra condition. We list a few such examples.

The general result is also called a **generalisation** of the special case. The process of generalisation is an extremely important part of mathematics.

Example: A square is a special case of a rectangle.

So, the formula for the area of a square ($A = a^2$) is a special case of the formula for the area of a rectangle ($A = ab$) obtained by putting $b = a$.

Similarly, the formula for the perimeter of a square is a special case of the formula for the perimeter of a rectangle.

Example: An isosceles right-angled triangle is a special case of a right-angled triangle.

So, from any theorem about all right-angled triangles, we can extract a special case that applies to isosceles, right-angled triangles.

Take a right-angled triangle ABC with $\angle C = 90°$; then we have $a^2 + b^2 = c^2$ (this is the Baudhāyana–Pythagoras theorem). Let us take a special case of this with $a = b$ (this corresponds to the triangle being isosceles). We get $a^2 + b^2 = c^2$, i.e., $c^2 = 2a^2$ and so $c = a\sqrt{2}$. The general theorem ($a^2 + b^2 = c^2$) has reduced to this special case ($c = a\sqrt{2}$) for isosceles right-angled triangles (with $a = b$).

Here is another example. Compare the following identities from algebra:

$$(a+b)^2 = a^2 + b^2 + 2ab,$$

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca.$$

Can you see that the first identity is a special case of the second one (put $c = 0$ in the second identity), and the second identity is a generalisation of the first one?

You will meet many examples of special cases and generalisation in later years.

Brahmagupta's Formula Generalises Heron's Formula

The similarity in appearance between Heron's formula and Brahmagupta's formula is not an accident. How? Brahmagupta's formula generalises Heron's formula. That is, we can think of Heron's formula as a special case of Brahmagupta's formula.

A triangle with sides a, b, c can be regarded as a 'special case' of a 4-gon ABCD whose fourth side d has zero length ($d = 0$). So we have AB = a, BC = b, CD = c, which means in effect that vertices A and D coincide.

(If the vertices represent planets, then this would be a case of two planets colliding with each other and forming a single large planet! This actually happened in the early history of the solar system.)

Since any triangle is cyclic (given any three points not in a straight line, we can draw a circle through them), this 4-gon is cyclic too. So, Brahmagupta's formula must apply to this 4-gon.

Let us see what emerges from this. We first compute the semi-perimeter of the 4-gon (keep in mind that $d = 0$):

$$s = \frac{1}{2}(a+b+c+0) = \frac{1}{2}(a+b+c).$$

We see that s is the same as the semi-perimeter of the given triangle. Now we apply Brahmagupta's formula:

$$\text{Area of triangle} = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

$$= \sqrt{s(s-a)(s-b)(s-c)} \quad \text{since } d = 0.$$

This is Heron's formula!

Brahmagupta's formula may thus be viewed as a generalisation of Heron's formula.

6.9 Squaring a Rectangle

In ancient times, to 'square a given shape' meant to 'construct a square equal in area to that shape'. The shape could be a rectangle or a triangle or a circle. Here, we show how the ancient Indian mathematician Baudhāyana squared a rectangle. The construction, from his *Śhulbasūtra* (800 BCE), shows how to square a rectangle with sides a units and b units where $a > b$. This means that we must construct a square with area *ab* sq. units. Here is a slightly simplified form of Baudhāyana's construction (Fig. 6.30).

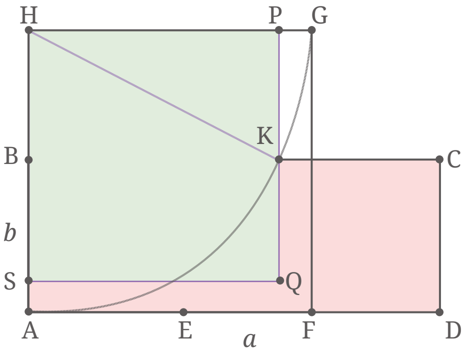


Fig. 6.30: Rectangle ABCD with AD = a, AB = b

- Given: rectangle ABCD with AD = a, AB = b where $a > b$.
- Locate E on AD so that AE = AB.
- Locate the midpoint F of ED.
- Draw square AFGH with side AF and vertex H on AB produced.
- Draw arc AG with centre H. Let arc AG cut side BC at K.
- Draw a line through K parallel to AH. Let it cut GH at P.
- Draw square HPQS with HP as side. Then square HPQS has the same area as rectangle ABCD.

Try to work out why this method works. You will find that it is a geometrical translation of the formula $\left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2 = ab$.

*Why does this method work?

Refer to Fig. 6.30. From $AE = b$ we get $\text{AF} = \frac{\text{AE}+\text{AD}}{2} = \frac{a+b}{2}$. Hence, $\text{HG} = \frac{a+b}{2} = \text{HK}$ as HK is a radius of the circle.

Next, $\text{BH} = \text{AH} - \text{AB} = \text{AF} - \text{AB} = \frac{a+b}{2} - b = \frac{a-b}{2}$.

Now consider the right-angled triangle HKP. Using the Baudhāyana–Pythagoras theorem we get

$$\text{HP}^2 = \text{HK}^2 - \text{BH}^2 = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2 =$$

$$\frac{a^2+2ab+b^2}{4} - \frac{a^2-2ab+b^2}{4} = \frac{ab}{2} + \frac{ab}{2} = ab.$$

Therefore square HPQS has the same area as rectangle ABCD.

Think and Reflect

What procedure would you use to square a given triangle? Here, the task is to construct a square whose area is equal to the area of some given triangle. Think carefully. How would you proceed?

Exercise Set 6.2

1. Find the area of triangle ADE in Fig. 6.31.

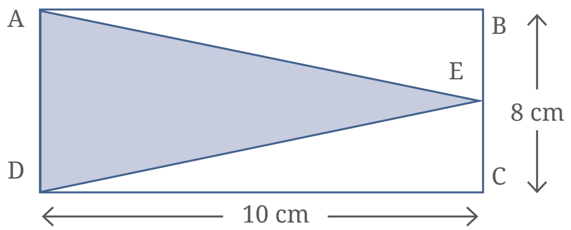


Fig. 6.31

2. The parallel sides of a trapezium are 40 cm and 20 cm. If its non-parallel sides are both equal, each being 26 cm, find the area of the trapezium.
3. Find the area of a triangle, given that its sides are 8 cm and 11 cm long, and its perimeter is 32 cm.
4. The sides of a triangular plot are in the ratio 3: 5: 7; its perimeter is 300 m. Find its area.
5. One diagonal of a rhombus is twice as long as the other diagonal. If the rhombus has area 128 cm^2, find the length of the shorter diagonal.
6. ABCD is a parallelogram. P and Q are any two points on side AB. What can you say about the ratio area (ΔPCD): area (ΔQCD)?
7. O is any point on the diagonal PR of a parallelogram PQRS. Prove that the areas of triangles PSO and PQO are equal.
8. If the mid-points of the sides of a 4-gon (also known as a quadrilateral, but we prefer to call it a '4-gon') are joined in order, prove that the area of the parallelogram thus formed will be half of the area of the given 4-gon. (You may wonder whether

the 4-gon thus formed is always a parallelogram, and if so, why? These questions will be tackled and answered in the chapter on quadrilaterals.)

9. In ΔABC, the midpoint of BC is D (Fig. 6.32). Median AD is drawn. P is any point on AD. Show that area (ΔABP) = area (ΔACP).

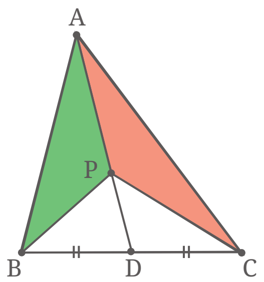


Fig. 6.32

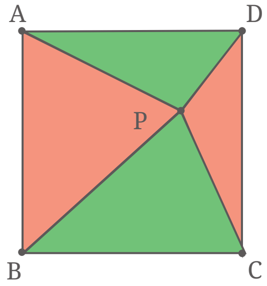


Fig. 6.33

10. Given a square ABCD, let P be a point within it. Join PA, PB, PC, PD (Fig. 6.33). What is the ratio of the areas of the red region (ΔPAB and ΔPCD) and the green region (ΔPBC and ΔPDA)?

11. In ΔABC, D is the midpoint of AB. P is any point on BC, and Q is a point on AB such that CQ || PD. PQ is joined (Fig. 6.34). Prove that Area (ΔBPQ) = $\frac{1}{2}$ Area (ΔABC).

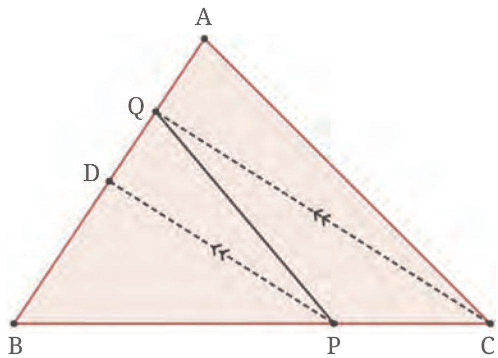


Fig. 6.34

6.10 Area of a Circle

A circle is an extremely basic shape; it is easy to draw if we have a rope and a peg to which the rope can be tied. Human beings have used the circular shape in their settlements, buildings and tools for thousands of years.

Think and Reflect

Why were human beings so fond of using circular shapes? Was this only for practical reasons, or could there have been other reasons too? What kinds of uses have human beings found for the circular shape?

Early on, human beings faced the challenge of finding the area A enclosed by a circle. They may have needed, for example, to work out how much grain can be kept in a cylindrical tower (which has a circular cross section), or how much area is occupied by a circular garden or a circular building. They needed such data for planning out their cities, for tax purposes, etc.

Early societies understood very well that the area A must be proportional to the **square** of the circumference C. Let's see why this is so.

- Suppose we are given a square. Let its side be a. It has perimeter P = $4a$ and area A = a^2. So the ratio $P^2 : A = (4a^2) : a^2 = 16a^2 : a^2 = 16 : 1$. So the ratio $P^2 : A$ is 16 : 1 for all squares; the ratio does not depend on the size of the square—it is the same for all squares.
- Consider an equilateral triangle with side a. Its perimeter is P = $3a$ and its area is $A = \frac{\sqrt{3}}{4}a^2$. So $P^2 : A = (3a)^2 : \frac{\sqrt{3}}{4}a^2 = 9a^2 : \frac{\sqrt{3}}{4}a^2 = 36 : \sqrt{3}$.

So the ratio $P^2 : A$ is $36 : \sqrt{3}$ for all equilateral triangles; the ratio does not depend on the size of the equilateral triangle—it is the same for all equilateral triangles.

- Similarly, for various shapes, as we change the scale, the ratio $P^2 : A$ stays fixed. The value of the ratio depends only on the shape, not on its scale.

Such reasoning suggests that for a circle too, if C = circumference and A = area, the ratio $C^2 : A$ must be some fixed constant. But what is this constant?

Well before 1500 BCE, the Babylonians had found through actual measurement that the constant is close to 12; that is, $C^2 : A \approx 12 : 1$. So, their formula for the area of a circle was $A \approx \frac{C^2}{12}$.

Around 1500 BCE, the ancient Egyptians came up with another such formula which resembles the one described above but is much more accurate: $A \approx \left(\frac{8d}{9}\right)^2$ where d is the diameter of the circle. Since $d = 2r$, this may also be written as $A \approx \left(\frac{64}{81}\right)4r^2$, i.e., $A \approx \left(\frac{256}{81}\right)r^2$.

Amazingly, the same formula appears in the Baudhāyana Śhulbasūtra (800 BCE) via a geometric method for constructing a square with (approximately) the same area as a circle.

The Familiar Formula in Use Today

The ancient Greeks also knew that $\frac{A}{r^2}$ is some constant—but did not know what that constant was. Two different civilisations approximated it to be $\frac{256}{81}$.

Finally, in *c.* 250 BCE, Archimedes showed that the constant is exactly π (i.e., the same constant that occurs in the formula for perimeter of a circle!); so, $A = \pi r^2$. He expressed the formula this way: "The area of a circle is equal to the area of a right angle triangle with sides equal to the radius and the circumference of the circle". That is,

$$\text{Area of circle} = \frac{1}{2} \times \text{circumference} \times \text{radius} = \frac{1}{2} \times 2\pi r \times r = \pi r^2.$$

The proof given by Archimedes is very clever. It is based on a property shared by all regular polygons: The area enclosed by a regular polygon is equal to half the perimeter × the radius of the circle that fits tightly within the polygon (Fig. 6.36). (The proof of this formula is an extension of the proof of the formula of the area of a triangle in terms of the radius of its incircle.)

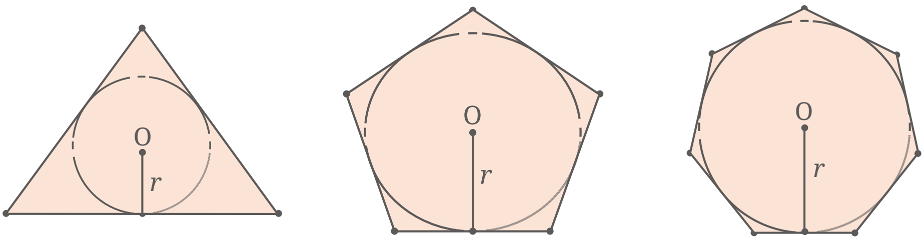
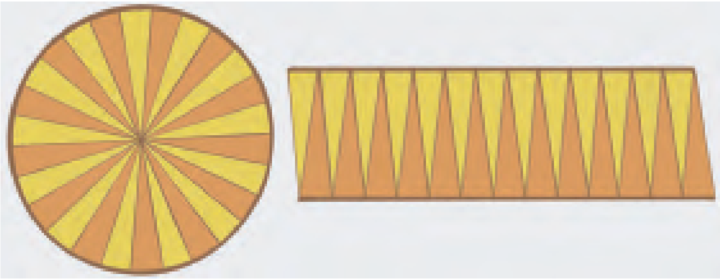


Fig. 6.36: Area of a regular polygon = $\frac{1}{2}$ *× perimeter of the polygon × radius*

Archimedes did a 'thought experiment'; he asked himself what happens if the number of sides of the regular polygon gets larger and larger. Working out the details, he arrived at the result $A = \pi r^2$.

Here perhaps is the most visual and easy-to-understand explanation for why the area of a circle of radius r has area πr^2. This beautiful visual explanation for the area of the circle was first discovered by Nīlakaṇṭha Somayājī (*c.* 1500) in his commentary on the *Āryabhaṭīya*:



(A) (B)

Fig. 6.37: The slices can be rearranged to form a parallelogram-like structure

As the slices become smaller and smaller, the arcs in Fig. 6.37B become more and more closer to a line. This makes the figure more and more closer to a parallelogram with

base = half the circumference (why?) = πr,

height = radius r.

This gives a way to argue that the area of the circle is equal to the area of a parallelogram!

Area of the circle = Area of the parallelogram = base × height = πr^2.

6.10.1 Area of Sector of a Circle

A sector of a circle is the region bounded by an arc and the two radii containing the endpoints of the arc.

The area within a sector of a circle may be found the same way that we found the length of a circular arc. Examine Fig. 6.39 and Fig. 6.40.

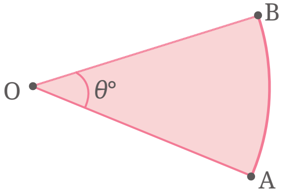


Fig. 6.38: Sector of a circle

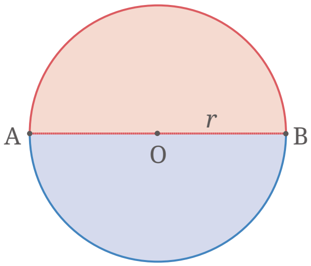


Fig. 6.39: Area of a semi-circular disc

By symmetry, we see that the area of a semicircular disc is $\left(\frac{180}{360}\right)^{th} = \frac{1}{2}$ of the area of the circle, i.e., $\frac{1}{2}\pi r^2$.

We could use the reflection symmetry of the circle to understand this result.

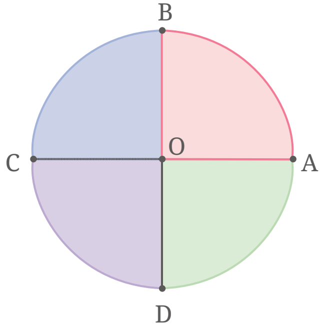


Fig. 6.40: Area of a quarter circular disc

By symmetry, we see that the area of a quarter circular disc is $\left(\frac{90}{360}\right)^{th} = \frac{1}{4}$ of the area of the circle, i.e., $\frac{1}{4}\pi r^2$.

We could also use the quarter-turn symmetry of the circle to understand this result.

Examining the above, we immediately obtain the desired formula for a general sector:

Formula for the area of a sector of a disc in terms of the angle it subtends at the centre of the circle

If the arc is AB, and it subtends an angle of θ^c at the centre O of the circle (see Fig. 6.38), then the area of the sector is

$$\pi r^2 \times \frac{\theta^\circ}{360^\circ}.$$

Note how we have used the rotational symmetry of a circle in deducing these formulas.

Related to a sector, there is a part of the circular region called a segment. A **segment** of a circle is the region bounded by an arc of the circle and the chord joining the endpoints of the arc.

Exercise Set 6.3

Unless stated otherwise, use the approximation $\frac{22}{7}$ for π.

1. Find the area of a sector of a circle with radius 7 cm if the angle of the sector is 60°.

2. Find the area of a quadrant of a circle whose circumference is 44 cm.

3. The length of the minute hand of a clock is 7 cm. Find the area swept by the minute hand in 10 minutes.

4. A chord of a circle of radius 10 cm subtends 90° at the centre. Find the area of the corresponding: (i) minor sector (that subtends 90° at the centre), and (ii) major sector (that subtends 270° at the centre). (Use $\pi \approx 3.14$.)

5. A chord of a circle of radius 15 cm subtends an angle of 60° at the centre of the circle. Find the areas of the corresponding minor and major segments of the circle. (Use $\pi \approx 3.14$ and $\sqrt{3} \approx 1.73$.)

6. A car has two wipers which do not overlap. Each wiper has a blade of length 28 cm and sweeps through an angle of 120°. Find the total area cleaned at each sweep of the blades.

*7. A chord of a circle of radius r subtends an angle of 60° at the centre of the circle. Show that the area of the corresponding minor segment of the circle is equal to $\pi r^2\left(\frac{1}{6} - \frac{\sqrt{3}}{4}\right)$.

*8. An equilateral triangle is inscribed in a circle of radius r. Show that the ratio of the area of the triangle to the area of the circle is equal to $\frac{3\sqrt{3}}{4\pi} \approx 0.413$.

*9. A square is inscribed in a circle of radius r. Show that the ratio of the area of the square to the area of the circle is equal to $\frac{2}{\pi} \approx 0.637$.

*10. A hexagon is inscribed in a circle of radius r. Show that the ratio of the area of the hexagon to the area of the circle is equal to $\frac{3\sqrt{3}}{2\pi} \approx 0.827$. Can you see why the answer is exactly twice the answer to Question 8?

End-of-Chapter Exercises

In the problems below, unless stated otherwise, use the approximation $\frac{22}{7}$ for π.

1. Identities in algebra can sometimes be shown as area relationships. For example:

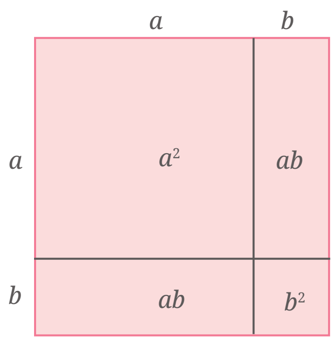


Fig. 6.41: Area model of an identity

The figure shown corresponds to the identity

$$(a + b)^2 = a^2 + 2ab + b^2.$$

Do you see how?

Draw figures corresponding to the identities $(a + b)(a - b) = a^2 - b^2$ and $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$.

2. An isosceles triangle has perimeter 40 cm; the equal sides are 15 cm each. Find the area of the triangle.

3. An isosceles triangle has base 10 cm, and its area is 60 cm^2. What are the lengths of the equal sides?

4. The area of a right-angled triangle is 54 sq. cm. One of its legs has length 12 cm. Find its perimeter.

5. The sides of a triangle are in the ratio 2: 3: 4, and its perimeter is 45 cm. Find its area.

6. The sides of a triangle have lengths 7 cm, 24 cm, 25 cm. Find the area of the triangle in two different ways.

7. If the wheel of a bicycle has a diameter of 60 cm, find how far a cyclist will have travelled after the wheel has rotated 100 times.

8. Find the area of a quadrant of a circle whose circumference is 66 cm.

9. The wheel of a car has an outer radius of 28 cm. Calculate how far the car travels after one complete turn of the wheel, and how many times the wheel turns during a journey of 1 km.

*10. Two rectangles have the same area and the same perimeter. Does this mean that they are congruent to each other?

11. You know that the area of a parallelogram is base × height. Using this and the figure, show that the area of a trapezium is half the sum of the parallel sides × height, i.e., $\frac{1}{2}(a + b)h$.

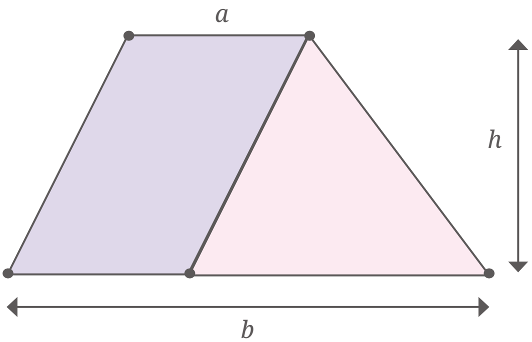


Fig. 6.42: Trapezium: sides a and b, height h

12. By dividing a trapezium into two triangles show that its area is, half the sum of the parallel sides multiplied by the height (the same formula as the one given above).

13. Show how we can use two identical copies of a trapezium to make a parallelogram. How will this give us the formula for the area of a trapezium?

14. Show that the area of a kite is half the product of its diagonals. Show this: (i) using algebra, and (ii) using geometry.

15. Three problems about fitting congruent shapes together:

    (i) Rectangle ABCD has sides a, b, and rectangle PQRS has sides $2a$, $2b$. Show that PQRS has 4 times the area of ABCD. Does this mean that 4 copies of rectangle ABCD will fit into rectangle PQRS? Check and see!

(ii) ΔABC has sides a, b, c, and ΔPQR has sides $2a$, $2b$, $2c$. Show that ΔPQR has 4 times the area of ΔABC. Does this mean that 4 copies of ΔABC will fit into ΔPQR? Check and see!

(iii) ΔABC has sides a, b, c, and ΔPQR has sides $3a$, $3b$, $3c$. Show that ΔPQR has 9 times the area of ΔABC. Does this mean that 9 copies of ΔABC will fit into ΔPQR? Check and see!

*16.

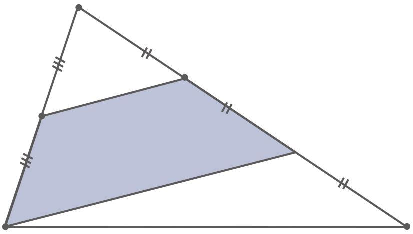


Fig. 6.43: What fraction of the triangle is shaded?

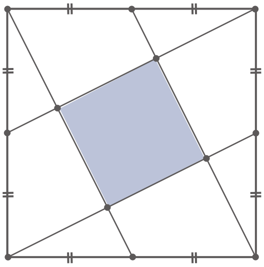


Fig. 6.44: What fraction of the square is shaded?

17.

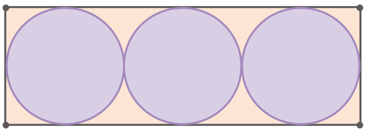


Fig. 6.45: What fraction of the rectangle is covered by the circles?

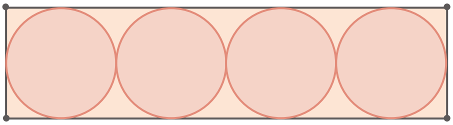


Fig. 6.46: What fraction of the rectangle is covered by the circles?

18. Use the above to make a conjecture about the area occupied by circles fitted into a rectangle in the manner shown. Test your conjecture for particular cases: 10 circles; 20 circles; 50 circles. Then prove your conjecture!

*19.

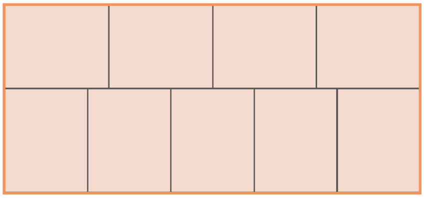


Fig. 6.47: Nine identical rectangles stacked together

The figure shows nine identical rectangles fitted together to make a large rectangle whose area is 72 cm^2. Find the perimeter of each small rectangle.

*20.

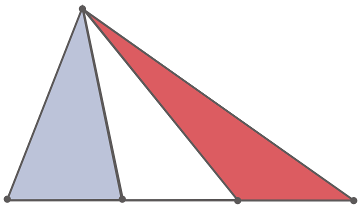


Fig. 6.48: Lines from a vertex to the points of trisection of the opposite side

Show that the areas of the shaded blue triangle and the shaded red triangle are equal.

Find a way of cutting up the blue triangle into some number of pieces and rearranging the pieces to cover the red triangle.

*21.

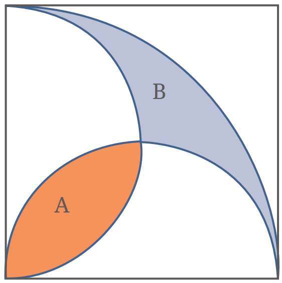


Fig. 6.49: A quarter circle and two semicircles

The figure shows a quarter circle in a square. Its centre is at one vertex, and it passes through two adjacent vertices. There are two semicircles on two adjacent sides as diameters. They create the shaded regions A and B .

Show that A and B have equal area.

*22.

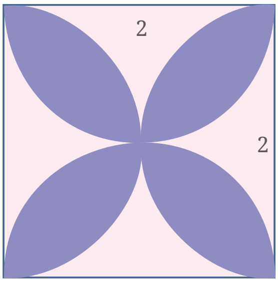


Fig. 6.50

In Fig. 6.50, four semicircles have been drawn within the given square whose side is 2 units. The centres of these semicircles are the midpoints of the sides. They create a 4-petalled flower (shown in blue). Find the perimeter and the area of this flower.

*23.

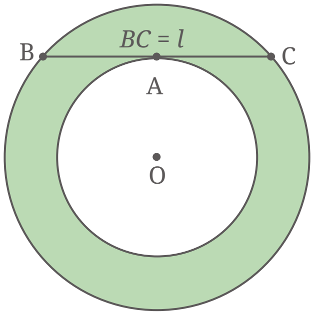


Fig. 6.51

In Fig. 6.51 we see two concentric circles with a common centre O. A chord BC of the larger circle is drawn, touching the smaller circle at A. The length of BC is l. Show that the area of the green region enclosed between the two circles is $\frac{1}{4}\pi l^2$.

*24. In Fig. 6.52, semicircles have been drawn on all the sides of a right-angled triangle as shown. Show that Area (A) + Area (B) = Area (C).

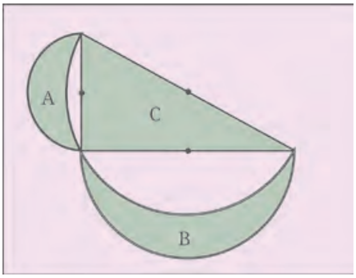


Fig. 6.52

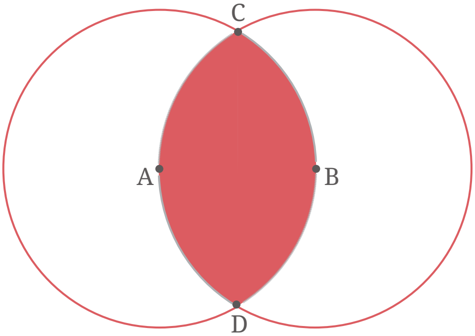


Fig. 6.53: Two congruent circles, radius r

*25. Fig. 6.53 shows two circles passing through each other's centres. Find the area of the region enclosed by the two circles in terms of the common radius *r*.

*26. In Fig. 6.54, we see three triangles within a rectangle.

The areas of the triangles are *A*, *B*, *C*, as marked. Show that the area of the rectangle is

$$\frac{2(A+C)(B+C)}{C}.$$

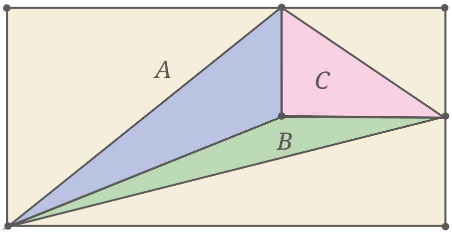


Fig. 6.54

*27. In the figure we see two shaded regions formed by a quarter circle, a semicircle, and a triangle.

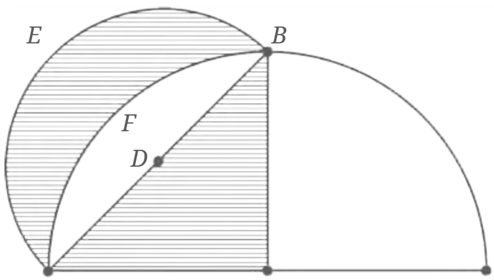


Fig. 6.55

Show that the areas of the two shaded regions are equal.

Chapter Summary

- π is the constant circumference to diameter ratio for all circles, and is approximately equal to $\frac{22}{7}$ or 3.14.
- The circumference of a circle is given by $C = 2\pi r$, where r is the radius of the circle.
- The arc length of a circle is given by $l = 2\pi r \times \frac{\theta°}{360°}$, where θ is the central angle.
- The area of a triangle is given by $A = \frac{1}{2}$ base times height.
- Heron's formula for the area of a triangle in terms of its sides a, b, c:

  area = $\sqrt{s(s-a)(s-b)(s-c)}$. where $s = s = \frac{1}{2}(a+b+c)$.
- The area of a circle is given by $A = \pi r^2$.
- Estimates for π: Archimedes: $3\frac{10}{71} < \pi < 3\frac{1}{7}$; Chongzhi: $\pi \approx \frac{355}{113}$; Āryabhaṭa: $\pi \approx 3.1416$; Mādhava's exact formula: $\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$.
- π is an irrational number.
- The area of a sector of a circle is given by area = $\pi r^2 \times \frac{\theta°}{360°}$, where θ is the central angle.
- Brahmagupta's formula for the area of a cyclic quadrilateral in terms of its sides a, b, c, d:

  area $= \sqrt{(s-a)(s-b)(s-c)(s-d)}$. where $s = \frac{1}{2}(a+b+c+d)$.

7 The Mathematics of Maybe: Introduction to Probability

7.1 What is Probability?

Probability is a type of measurement, similar to how we measure quantities like length, area, or volume. However, instead of measuring physical quantities, probability is used to measure the likelihood of **events**. Specifically, it helps us express how confident or certain we are that a particular event will occur. For example, you may ask your friend:

- Is it going to rain today?
- Will our school win the inter-school hockey match tomorrow?
- Will I be chosen in the monthly lucky draw to perform at the school assembly? [The names of all the students in school are written on slips of paper and one slip is randomly selected.]

These events are examples of **random** events. We know the possible outcomes (either it will rain today or it will not; our school team will either win, draw or lose the hockey match; one student will be chosen to perform at the school assembly), but we do not know in advance which one will definitely occur. That is, there is an element of **chance** or **randomness** involved every time such an event takes place.

Can we predict these outcomes with 100% certainty? One could respond to these questions with words such as **impossible** or **certain**, or using phrases such as **less likely**, **more likely** or **equally likely**. This decision is based on different kinds of **evidence** that have been gathered. For example, one friend might say, “The sun is shining brightly, so it’s unlikely to rain today”, while another might observe, “It’s very hot, which makes me think it could rain later”. In both cases, they are predicting rainfall based on how they interpret the present weather conditions. This is the **subjective probability** given to the event of today’s rainfall.

As you can see from the above example, probability deals with **uncertainty** or **chance**. One key feature of our increasingly complicated society is that we must deal with questions that have no fixed answer but rather one or more possibilities for the answer. Thus, understanding how to **objectively** estimate the probability of events is crucial in many aspects of life.

In this chapter, we will learn how we can measure probability more objectively, i.e., the ways of **collecting evidence** that can be used for an objective estimate of likelihood. But first, we must understand a few terms and ideas such as randomness and the probability scale.

7.1.1 What is Randomness?

Randomness refers to a situation or action (like the tossing of a coin or the rolling of a die) where you cannot predict exactly what will happen. Although you may know all the possible outcomes, you cannot say which one will definitely occur. For example,

- **Tossing a coin:** You know it could be heads or tails, but you cannot be sure which one will come up in a single toss.
- **Rolling a die:** You know the possible outcomes are 1, 2, 3, 4, 5, or 6, but you do not know which number will appear on a particular roll.

These observations (commonly called **experiments** or **trials** in probability) are called random because of their unpredictability. The lucky draw example mentioned earlier is an example of a random observation or experiment, as the outcome cannot be predicted in advance, and each student has an equal chance of being selected. All you can say is what could happen, not what will happen.

Random Observations: A random experiment is something you can repeat (like tossing a coin), where every time you do it, the result might be different and you cannot know the outcome in advance.

Think and Reflect

Such unpredictability can be useful sometimes! For example, in a cricket match, the fact that a coin is tossed to decide which team will bat first is considered to be a fair method. Can you explain why?

Probability is the area of mathematics that studies randomness and how likely a specific outcome is to happen in a random situation. For example, when you toss a coin, the probability for heads is $\frac{1}{2}$, and for tails is $\frac{1}{2}$, because each is **equally likely**. In a random experiment, every outcome has a chance to occur, but you can only determine the likelihood, not the exact result.

Have you wondered what makes an event like rain random? An event like rain is considered random because it depends on many complex factors in the atmosphere (such as temperature, humidity, wind patterns, and pressure) and is so sensitive to these factors that it is impossible to predict it with total certainty. Thus, it is impossible to know with absolute certainty whether it will rain on a specific day. Randomness in rain means that while the exact timing and location of rainfall cannot be predicted perfectly, we can estimate the likelihood or probability of rain in different locations based on patterns and probabilities derived from data.

Think and Reflect

Ask your friend to predict the outcome of a ₹1 coin you toss. Do you see that your friend could guess heads or tails but could not know for certain? That's randomness! All possible results are known, but each individual try is unpredictable.

7.1.2 The Probability Scale

Probability is measured on a scale from 0 to 1 to indicate the likelihood of the occurrence of an event. If the probability of your school winning the hockey match is 0.75, that means there is a 75% chance your school will win. This means that it is **more likely** than not that your school will win the match. If the probability of your school winning the hockey match is 0.5, that means there is a 50% chance your school will win. This means that it is **equally likely** that your school will win or lose the match. On the other hand, if the probability is 0, it would mean winning is **impossible** (for example, winning without playing), and if the probability is 1, it would mean your school is certain to win. The probabilities of most events fall somewhere strictly between 0 and 1, expressing how likely they are to happen.

Imagine a deck of six cards where the number of purple and green cards is unknown. The probability of picking a purple card from the deck (see Fig. 7.1) can range from impossible (if there are no purple cards) to certain (if all six are purple). As the number of purple cards increases, the likelihood of selecting a purple card moves smoothly along the **probability scale** from less likely, to even chance, to more likely. This scale helps us understand and compare how likely different events are, just like using a number line!

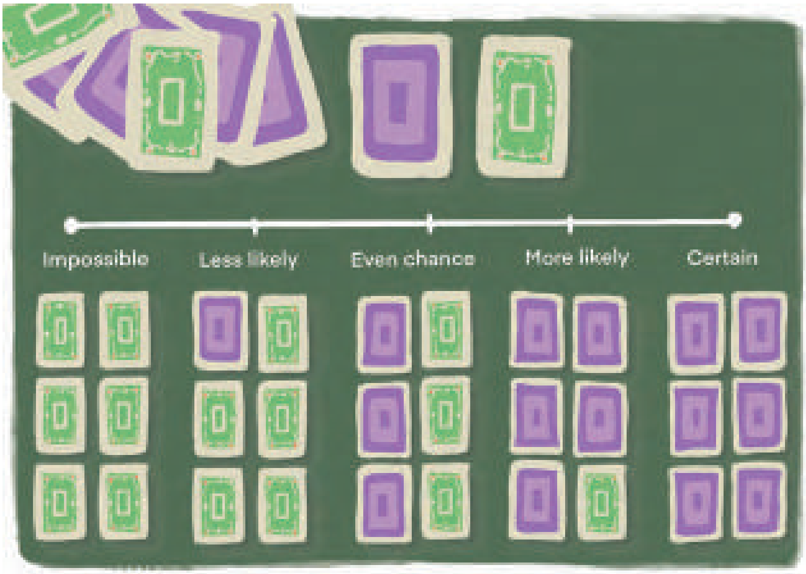


Fig. 7.1: Probabilities of picking a purple card from a deck on the probability scale

Here are a few examples of events and the probabilities of their occurrence. Look at the examples against the probability scale in Fig. 7.1 to understand how probability is measured on a scale.

Event	What it means
Getting a number greater than 6 on a die	**Impossible:** Dice only have numbers from 1 to 6.
Rolling a 3 on a standard die	**Less likely:** But not impossible as one face of the die is 3.
Flipping a coin and getting heads	**Even chance:** Heads or tails are equally likely.

Drawing any number from 2 to 10 from a deck of 52 cards	**More likely:** There are 36 cards in a deck of 52 cards with numbers from 2 to 10.
Choosing a red sweet from a bag of all red sweets	**Certain:** All the sweets are red.

Exercise Set 7.1

1. Rank the following events on a scale from 0 (Impossible) to 1 (Certain). Label each event: Impossible, less likely, equally likely (even chance), more likely, certain. Give reasons why you gave each event its ranking.
   (i) The next Monday will come after Sunday.
   (ii) It will snow in Mumbai in July.
   (iii) An elephant will walk through your classroom today.
   (iv) You will greet at least one friend at school tomorrow.

7.2 Measuring Probability Objectively

There are two main ways of estimating the probability of an event objectively:

1. **Using evidence from experience:** This involves collecting data either by performing an experiment multiple times or by analysing statistical data from past observations. In both cases, the probability is estimated by calculating the relative frequency of the event based on the collected evidence. We call this e**xperimental probability**.

2. **Using theoretical methods:** This approach assumes that all **possible outcomes** are equally likely (when there is no reason to believe that any one outcome is more likely than another). The probability is then determined by reasoning about the number of favourable outcomes in relation to the total number of possible outcomes. We call this **theoretical probability**.

7.2.1 Experimental Probability: Performing Observations or Experiments

Another way of obtaining objective estimates for probabilities is to perform an experiment. In an experiment, the event is a result of the experiment and is called an **outcome**. The set of all possible outcomes in an experiment is called the **sample space**.

The sample space may be listed within brackets, separated by commas as shown below.

Example 1: Experiment: A coin is tossed
Possible Outcomes: Heads (H) or Tails (T) (See Fig. 7.2)
Sample Space: {H, T}

Experiment: A die is rolled
Possible Outcomes: The top side of the die shows 1, 2, 3, 4, 5, or 6 dots (See Fig. 7.3)
Sample Space: {1, 2, 3, 4, 5, 6}

Fig. 7.2: Heads and tails of a coin

Fig. 7.3: A 6-sided die

$$\text{Experimental Probability} = \frac{\text{Number of times the event occurred}}{\text{Total Number of trials}}.$$

Example 2: Suppose you roll a die 50 times, and it lands on a 4 exactly 8 times.

Experimental probability of rolling a 4 is $\frac{8}{50} = 0.16$ or 16%.

We also say that the **relative frequency** of rolling a 4 is $\frac{8}{50} = 0.16$.

Relative frequency helps you understand probability based on actual data, rather than theoretical predictions. It is especially useful in statistics and data analysis when you are working with observed outcomes.

Did you know? The game of Snakes and Ladders originates in ancient India. It evolved from a traditional dice game called **Jñān-Chaupaḍ**, which was used as an educational tool. Jñān-Chaupaḍ aimed to convey moral lessons, with each ladder symbolising a virtue and each snake representing a vice. Through gameplay, it illustrated the consequences of good and bad behaviour. Various versions of Jñān-Chaupaḍ have been found in different parts of India.

Fig. 7.4: A Jain Jñān-Chaupaḍ game on cloth, National Museum (India, 19th century)

7.2.2 Theoretical Probability

Theoretical probability studies the likelihood of an event happening based on all possible outcomes being **equally likely**. It is what we expect to happen in an ideal, **perfectly fair situation**. In this scenario, no experiment or data is required. Theoretical probability is usually denoted as P (Event) or P (Outcome).

$$\textbf{Theoretical Probability (P)} = \frac{\text{Number of favourable outcomes}}{\text{Number of possible outcomes}}.$$

Example 3: If you roll a standard 6-sided die, what is the theoretical probability of getting a 4?

Solution:

Number of favourable outcomes = 1 (only the number 4)

Number of all possible outcomes = 6 (the numbers 1 through 6)

$$\text{P (rolling a 4)} = \frac{1}{6} = 0.1666... \approx 0.167 \text{ or } 16.7\%.$$

Example 4: A letter is picked at random from the word 'PROBABILITY'. What is the probability of picking the letter B?

Solution:

Number of favourable outcomes = 2 (there are 2 Bs in the word PROBABILITY)

Number of all possible outcomes = 11 (The number of letters in the word PROBABILITY)

P (Picking the letter B) = $\frac{2}{11}$ = 0.1818... ≈ 0.182 or 18.2%.

7.2.3 Analysing Statistical Data Using Probability

The way of collecting evidence through statistical data is used extensively in the world of business for marketing, forecasting sales, insurances and in science and social science research.

Example 5: Suppose you anonymously collect information regarding the favourite fruit of 50 students in your class. Let us assume that the results are: 20 students like mango, 15 students like apples, 10 students like bananas, and 5 students like grapes.

Let us play a game! Suppose we randomly pick one student from the class and try to guess their favourite fruit. What's the probability that the student's favourite fruit is mango? Statistical probability is based on collected data. So, a reasonable estimate that the student's favourite fruit is mango is

Probability of mango as the favourite fruit

$$= \frac{\text{Number of students who like mango}}{\text{Total number of students}} = \frac{20}{50} = 0.4.$$

So, there is a 0.4 or 40% chance that a randomly chosen student in the class likes mango.

Now assume that you want to buy fruits for the entire school depending on the students' favourite fruits. You would need to estimate the quantity of each type of fruit you need to purchase for the entire school of, say, 1500 students. It is usually impractical to collect data from all students or the entire **population**. The application of statistical probability in the real world is usually based on evidence collected from a **sample**. The population, i.e., the total number of students in the school, is 1500, and we collected evidence from a sample of 50 students from one class. An estimate of the probability that mango is the favourite fruit was calculated to be 0.4. So, that would mean that we need to buy approximately 600 (40% of 1500) mangoes, while the remaining would be other fruits — apples, bananas and grapes. If we want to be more confident about our estimate, we would choose a larger sample, for example, by asking 100 students, and make sure it is more representative, such as including students from different classes or grades. This process is called **sampling** in statistics.

LEARN MORE ABOUT SAMPLING

Want to learn more about sampling? What we have learned so far is just the beginning!

When collecting data through samples, there are other important things to think about — like how big your sample should be and how to make sure your sample truly represents the whole population. The size of your sample and making sure it's fair and not biased can help your results be more accurate.

If you are interested and want to explore more about why sample size and representation matter in statistics, you can read more about it in print or digital resources.

Summarising, **experimental probability** is based on actual data collected from trials, not on assumptions. **Theoretical probability** relies on the assumption of equally likely outcomes (a perfectly fair situation) and does not use experimental data. Even in perfectly fair situations, experimental probability can differ from theoretical probability, especially when the number of trials is small. As the number of trials increases, the experimental probability tends to get closer to the theoretical probability — this is known as the **Law of Large Numbers**.

Think and Reflect

If I have rolled a 4 on a die 8 times in succession, the probability of rolling a 4 again is still only ≈ 0.16 (assuming the die is fair). Probability does not tell you what will happen next but predicts what will happen in the long run.

GAMBLER'S FALLACY

Many people think that if something random happens many times in a row (like flipping a coin and getting heads six times), then the opposite outcome (tails) is more likely to happen next. But actually, the coin does not keep track of what happened before. Every time you flip, the chance of heads or tails stays exactly the same — like starting afresh. The coin has no memory, so past flips do not change what happens next. This common misunderstanding is called **Gambler's Fallacy**.

For example, imagine that you are flipping a fair coin. It comes up heads six times in a row. You might feel that on the next flip you are sure to get a tail, because tails haven't come up for a while. However, the probability of getting tails on the next flip is still 50% or $\frac{1}{2}$, exactly the same as before. The coin has no memory of past flips!

Example 6: Let us say you are playing Snakes and Ladders, and you are rolling a fair 6-sided die to move.

You have just rolled the die three times in a row, and each time you got a 6.

Now, you think: 'I have already rolled three 6s — there is no way I will get a 6 again on the next roll!'

This thinking is Gambler's Fallacy — because each roll of the die is an independent event. The probability of rolling a 6 is always: $\frac{1}{6} \approx 0.166$ or 16.6%.

It does not change based on what happened in previous turns.

Key Idea: In Snakes and Ladders, as in many games of chance, each roll of dice is independent. The Gambler's Fallacy tricks people into believing there is a pattern, when in fact, such randomness has no memory.

FAIR AND UNBIASED

When we speak of a coin, we assume it to be **'fair'**, i.e., it is symmetrical so that there is no reason for it to come down more often on one side than the other. We call this property of the coin as being **'unbiased'**. By the phrase 'random toss', we mean that the coin is allowed to fall freely without any bias or interference.

Exercise Set 7.2

1. A teacher mixes a large bag of sweets of different colours and randomly selects a sample of 30 sweets. She counts the number of sweets of each colour:
   10 red sweets | 8 green sweets | 7 yellow sweets | 5 blue sweets
   (i) Calculate the probability that a randomly picked sweet from the sample is green.
   (ii) If there are 600 sweets in total in the large bag, estimate how many are likely to be yellow, based on the sample results.
2. A survey is conducted at a school where a random sample of 40 students is asked about their favourite club. The responses are:
   14 students: Science Club | 11 students: Arts Club |
   9 students: Sports Club | 6 students: Debate Club
   Assume there are 800 students in the whole school.
   (i) What is the probability that a randomly chosen student from the sample prefers the Arts Club?
   (ii) Using the sample results, estimate how many students in the whole school are likely to prefer the Sports Club.
3. Toss a coin 20 times and record the result each time (heads or tails).
   (i) How many times did you get heads?
   (ii) How many times did you get tails?
   (iii) Calculate the experimental probability of getting heads.
   (iv) If you toss the coin once more, what is the probability of getting tails?
4. Toss a paper cup into the air 100 times. After each toss record whether the cup lands on its bottom, upside down on its top or on its side (See Fig. 7.5). Assign probabilities to the outcomes by using experimental probability.
5. What is the probability of getting an even number when rolling a fair 6-sided die?

Fig. 7.5: Paper cup landing positions (left to right)—bottom, top and side

6. Suppose you roll a 6-sided die 12 times and get a '3' three times.
   (i) What is the experimental probability of rolling a '3'?
   (ii) What is the theoretical probability of rolling a '3'?
   (iii) Why might these probabilities be different? What would you expect to happen if you roll the die 60, 600, or 6000 times?

7.3 Elements of Probability: Sample Spaces and Events

7.3.1 Sample Space

Recall that the sample space, denoted by S, is the list of all possible outcomes. Each possible outcome is called an element of the sample space.

- The sample space S must include **every possible outcome**.
- No outcome should be **listed more than once**.
- The number of elements in the sample space is called the **sample size** and is denoted by n(S).

Examples of Sample Space

1. The sample space for whether it rains tomorrow could be S = {Rain, No Rain} and the sample size n(S) = 2.
2. The sample space for a match, i.e., the possible outcomes for a team playing — could be S = {Win, Lose, Draw} and sample size n(S) = 3.
3. Tossing a Coin:
   - Experiment: Tossing a fair coin once.
   - Sample Space: S = {Heads (H), Tails (T)}
   - Sample Size = 2
4. Rolling a Die:
   - Experiment: Rolling a standard 6-sided die.
   - Sample Space: S = {1, 2, 3, 4, 5, 6}
   - Sample Size = 6
5. Tossing Two Coins:
   - Experiment: Tossing two coins simultaneously.
   - Sample Space: S = {HH, HT, TH, TT}
   - Sample Size = 4

Coin 1	Coin 2	Outcome
H	H	HH
H	T	HT
T	H	TH
T	T	TT

Think and Reflect

When we used the sample space {Rain, No Rain} in Example 1, we focused only on whether it will rain or not. However, if we want to include different amounts of rainfall like drizzle, light rain or heavy rain, we need to expand the sample space to {No Rain, Drizzle, Light Rain, Heavy Rain} so that it better matches the level of detail required for the question. It is important to ensure the sample space is detailed enough to suit the specific problem being studied.

7.3.2 Events

An **event** is any single possible result or combination of results that might happen when you perform a random action. It is like choosing particular outcomes from all the things that could possibly occur. An event is a subset of a sample space.

Examples of Sample Spaces and Events

1. Tossing Two Coins
   - Sample space: All possible results like Head–Head, Head–Tail, Tail–Head, Tail–Tail. S = Sample Space = {HH, HT, TH, TT}
   - Event: 'At least one coin shows Head.' E = {HH, HT, TH}.
2. Rolling a 6-sided Die
   - Sample space: All faces numbered 1 through 6. S = {1, 2, 3, 4, 5, 6}
   - Event: 'The number rolled is greater than 4.' E = {5, 6}.
3. Picking Fruit from a Basket
   - Sample space: Types of fruit you might pick, for example, Apple, Banana and Orange. S = {Apple, Banana, Orange}
   - Event: 'Picking a fruit that is yellow.' E = {Banana}.

Exercise Set 7.3

1. When a single 6-sided die is rolled, what is the total number of possible outcomes in the sample space?
2. For the following experiments write down the sample space S.

   (i) Rolling a die and tossing a coin together.

(ii) Choosing a random integer between – 5 and + 5.

(iii) A box containing 5 green and 7 red balls. One ball is drawn at random.

3. In a village fair, there are 3 popular snacks available: Samosa, Pakora, and Bhaji. For drinks, villagers can choose either Chai or Lassi.

   (i) List the sample space of all possible snack and drink combinations a person could choose at the fair.

   (ii) List the event 'Selecting Samosa as a snack.'

7.4 Tree diagrams

A tree diagram is a visual representation used to list all possible outcomes of a **multi-step experiment**. A multi-step experiment involves a series of independent trials. For example, tossing a coin two times, or rolling a dice three times are examples of multi-step experiments. Each branch of the tree represents a possible outcome, and branches split to show different paths for subsequent events.

A tree diagram is useful for

- Visualising multi-step experiments, where each path from start to end represents one complete outcome.
- Listing all outcomes of a sample space.

So far in this chapter, we have seen examples of single-step experiments. Let us look at an example of a multi-step experiment.

Example 7: Experiment: Toss a fair coin two times.

Tree Diagram (See Fig. 7.6):

From a single point draw a line to each of the possible outcomes of the first toss. From each of these outcomes draw two lines to each possible outcome. Do you see that the tree diagram shows 4 possible outcomes?

Hence the Sample Space

S = {HH, HT, TH, TT}.

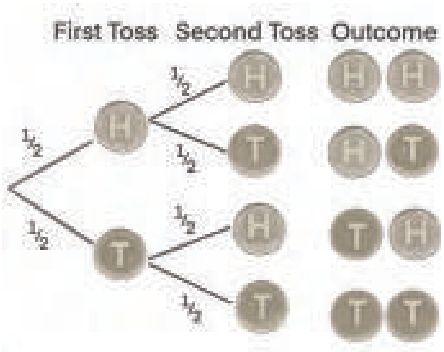


Fig. 7.6: Tree diagram showing possible outcomes when we toss a fair coin twice

Also, the theoretical probabilities of each outcome are mentioned on the branch representing that outcome.

$$\text{Theoretical Probability (P)} = \frac{\text{Number of favourable outcomes}}{\text{Number of possible outcomes}}.$$

Let us consider the event of getting Heads twice. This is represented by the outcome HH.

Probability of HH = $\frac{1}{4}$ = 0.25 or 25%.

Think and Reflect

Can you calculate the probability of getting one head and one tail?

Exercise Set 7.4

1. There are two fruit baskets A and B. Basket A has one apple and two oranges. Basket B has one banana and one mango. You randomly pick one fruit from each basket.
   (i) Draw a tree diagram showing all possible pairs of fruits.
   (ii) List the sample space.
   (iii) What is the probability of picking one apple and one banana?
2. Let us say that you have a box containing 3 red pens, 4 black pens and 2 green pens. You pick a pen (without looking) from the box and put it back. Then your friend does the same.
   (i) What are the possible outcomes of the pen colours? Can you draw a tree diagram representing the possible outcomes?
   (ii) Can you use the tree diagram to guess the probability that both you and your friend pick pens of the same colour?

End-of-Chapter Exercises

1. Fill in the blanks.
   (i) The probability of an impossible event is ______.

(ii) The set of all possible outcomes of a random experiment is called the __________.

(iii) The probability of an event that is certain to happen is _______.

(iv) Tossing a fair coin has a probability of ______ for getting heads.

2. In a survey of 50 students, 15 students said they liked football. The number of students who like football is 15, and the ________ (frequency/relative frequency) is __________ (fill in the fraction or decimal).

3. Which of the following experiments have equally likely outcomes? Explain.

   (i) A driver attempts to start a car. The car starts or does not start.

   (ii) Tossing a fair coin once.

   (iii) Rolling a fair 6-sided die.

   (iv) Choosing a marble randomly from a bag that contains 3 red marbles and 7 blue marbles.

   (v) A baby is born. It is a boy or a girl.

4. Write the sample space and calculate the probability based on the given information.

   (i) Two coins are tossed at the same time. What is the probability of getting at least one head?

   (ii) Ten identical cards numbered 1 to 10 are placed in a box. One card is drawn at random. What is the probability of drawing a card with an even number?

   (iii) A die is rolled once. What is the probability of getting a number greater than 4?

   (iv) A bag contains 3 red balls, 2 blue balls, and 1 green ball. One ball is picked at random. What is the probability that it is not red?

   (v) Three coins are tossed simultaneously. What is the probability of getting exactly two heads?

5. A bag has 3 candies: strawberry, lemon, and mint. One is picked at random. What is the probability of picking a strawberry candy?

6. A child has 2 shirts (one red and one blue) and 3 types of pants (jeans, khakis, and shorts). List all the possible combinations of outfits consisting of one shirt and one pair of pants. Display your answer in a table format.

7. A tyre company records distances before replacement in 1000 cases.

Distance (km)	Less than 4000	4001 to 9000	9001 to 14000	More than 14000
Number of cases	20	210	325	445

Find the probability that a randomly chosen tyre lasts:

(i) Less than 4000 km.

(ii) Between 4000 and 14000 km.

(iii) More than 14000 km.

8. The letters of the word 'PEACE' are placed on cards. Leela draws a card without looking.

(i) What is the probability that it is a P, E or C?

P	E	A	C	E

(ii) What is the probability that it is not an E?

*9. A game of chance consists of spinning an arrow (see Fig. 7.7.) which comes to rest pointing at one of the numbers 1, 2, 3, 4, 5, 6, 7, 8, and these are equally likely outcomes. What is the probability that it will point at

(i) 8?

(ii) An odd number?

(iii) A number greater than 2?

(iv) A number less than 9?

(v) A multiple of 3?

Fig. 7.7

*10. A basket contains 4 red balls and 5 blue balls. One ball is drawn and laid aside, and a second ball is drawn. Draw a tree diagram to represent the possible outcomes and probabilities. Use the tree diagram to answer the following questions.

(i) What is the probability of drawing a red ball and then a blue ball?

(ii) What is the probability of drawing 2 blue balls?

*11. I throw a pair of 6-sided dice. Write down an event that has a probability of 0 and an outcome that has a probability of 1.

*12. Write the sample space and calculate the probability based on the given information.

(i) Two dice are rolled. What is the probability that the sum is a prime number greater than 5?

(ii) A bag contains 4 red, 3 green, and 2 blue balls. Two balls are drawn without replacement. What is the probability that both are of different colours?

(iii) Three coins are tossed. What is the probability that the first coin shows heads and exactly two heads occur in total?

(iv) A four-digit number is formed using the digits 1, 2, 3, and 4 with no repetition. What is the probability that the number is even?

(v) A student takes a multiple-choice test with 3 questions, each having 4 options (A, B, C, D), with only one correct answer. What is the probability that the student guesses and gets exactly 2 answers correct?

*13. A box contains 4 balls numbered 1 to 4. Record a sample space using a tree diagram for the following experiments:

(i) A ball is drawn, and the number is recorded. Then the ball is returned, and a second ball is drawn and recorded.

(ii) A ball is drawn and recorded. Without replacing the first ball, the experimenter draws and records a second ball.

(iii) What are the sizes of these two sample spaces?

*14. List the elements of a sample space for the simultaneous tossing of a coin and drawing of a card from a set of 6 cards numbered 1 through 6.

*15. Three coins are tossed, and the number of heads is recorded. Which of the following lists is a sample space for this experiment? Why do the other lists fail to qualify as a sample space?

(i) {1, 2, 3}

(ii) {0, 1, 2}

(iii) {0, 1, 2, 3, 4}

(iv) {0, 1, 2, 3}

*16. Suppose you drop a dye at random on the rectangular region shown in Fig. 7.8. What is the probability that it will land inside the circle with a diameter of 1 m?

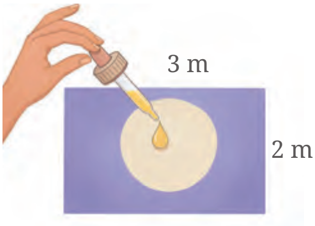


Fig. 7.8

Chapter Summary

- Probability is a measurement of the likelihood of an event.
- Probability is expressed on a scale from 0 (indicating impossibility) to 1 (indicating certainty), representing the likelihood of an event occurring. The probability of an event E is denoted as P(E), where $0 \leq P(E) \leq 1$.
- Experimental probability is determined by conducting an experiment and is calculated as:

  $$\text{Experimental Probability} = \frac{\text{Number of times the event occurred}}{\text{Total number of trials}}.$$

- Theoretical probability is based on all possible outcomes in a fair situation, assuming each is equally likely:

  Theoretical Probability of an event A is

  $$P(A) = \frac{\text{Number of favourable outcomes}}{\text{Number of possible outcomes}}.$$

- A sample space is the list of all possible outcomes of a random experiment. It is represented as a set S = {outcome_1, outcome_2 ..., outcome_n}.
- An event is any one or a group of possible outcomes from a random experiment. It is like picking results or outcomes from all that could happen.
- Tree diagrams help us to list and visualise all possible outcomes of a random experiment. They also help us calculate probabilities of events related to random experiments.

8 Predicting What Comes Next: Exploring Sequences and Progressions

8.1 Introduction to Sequences

We see patterns around us everywhere, be it in nature, in art, in music, in finance and in many other contexts in everyday life. Patterns help us make sense of the world and predict what comes next. In mathematics, sequences are special kinds of patterns formed by numbers or other objects arranged in a particular order. By understanding sequences, we can explore fascinating ideas about how numbers grow, shrink, or repeat and even use these ideas to solve real-life problems.

In this chapter, we shall explore patterns in sequences of numbers. We will then find rules to help us predict more numbers of the sequence. Let us begin by looking at some number sequences that you have already seen in Grades 6, 7 and 8.

1, 2, 3, 4, 5, 6, …	(Natural Numbers)
1, 3, 5, 7, 9, 11, …	(Odd Numbers)
1, 3, 6, 10, 15, 21, …	(Triangular Numbers)
1, 4, 9, 16, 25, 36, …	(Square Numbers)

The three dots … indicate that the sequence continues indefinitely.

Think and Reflect

Can you describe the pattern in each of the above sequences? Can you predict the next few numbers in these sequences?

Before we proceed, let us define a **sequence** as an ordered list of numbers where each number is a **term** of the sequence. Thus, in the sequence of square numbers, 1 is the first term, 4 is the second term, 25 is the fifth term and so on. Also, sequences may be finite or infinite. The sequences mentioned above are infinite. But the sequence 6, 12, 24, 48, 96 is a finite sequence of five terms. Can you think of other finite sequences that you see in your daily life?

As you already know, in the sequence of natural numbers, every number (or term) is one more than the previous number. In the sequence consisting of all the odd numbers, there is a difference of 2 between any two consecutive terms. In the sequence of triangular numbers, the difference between consecutive terms (among the first six terms) are 2, 3, 4, 5 and 6. We may rewrite the triangular numbers in the form of sums of natural numbers as 1 = 1, 3 = 1 + 2, 6 = 1 + 2 + 3, 10 = 1 + 2 + 3 + 4, and so on. Thus, each term of the triangular number sequence is the sum of the natural numbers up to that term. For example, 15, the fifth triangular number, is equal to 1 + 2 + 3 + 4 + 5. This is represented by the diagram in Fig. 8.1, where each triangular number is represented by a triangular array of dots. Can you draw the patterns for the next two terms of the sequence?

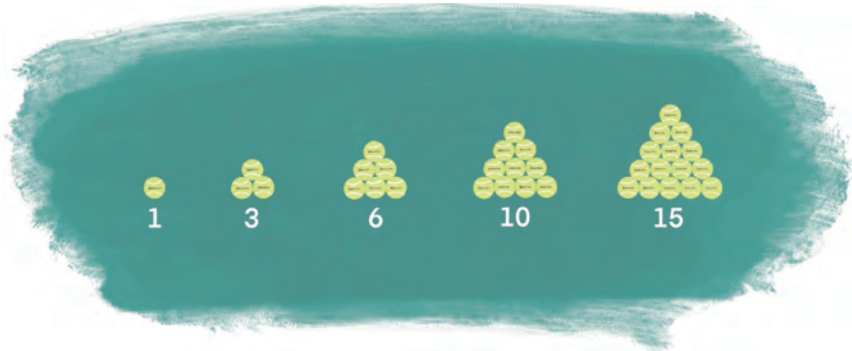


Fig. 8.1: The first five triangular numbers

Let us now shift our attention to the sequence of square numbers, 1, 4, 9, 16, 25, 36,... Note that the differences between consecutive terms (for the first six terms) are 3, 5, 7, 9 and 11. Also 1 = 1, 4 = 1 + 3, 9 = 1 + 3 + 5, 16 = 1 + 3 + 5 + 7, and so on. Each term in the square number sequence is the sum of the odd numbers up to that term.

This interesting relationship between the odd numbers and square numbers can be represented by the diagram in Fig. 8.2. Can you explain the relationship? You may recall some of these ideas from Grade 6, Chapter 1!

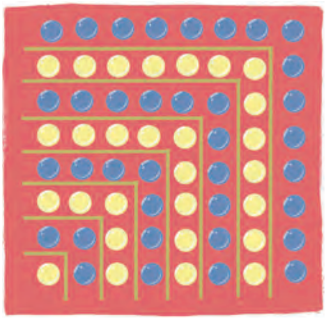


Fig. 8.2: Square numbers and odd numbers

Exercise: Consider the sequence 1, 4, 7, 10, 13, ... Can you predict the next four terms? Can you derive the first 10 terms of the sequence obtained by adding all the terms up to a given term of this sequence? (**Hint:** The first term is 1. The second term is 1 + 4 = 5, the third term is 1 + 4 + 7 = 12, and so on.)

In some sequences (like the ones described till now), the terms follow a certain pattern, a rule or order. To describe these rules it is helpful to have a convenient notation to describe a sequence. We can use t_1 to represent the first term of a sequence, t_2 to represent the second term, and so on. In this notation, the subscripts match the term numbers. Thus, for the sequence of odd numbers, $t_1 = 1$, $t_2 = 3$, $t_3 = 5$, $t_4 = 7$ and so on. This notation helps to connect the position of the term to the actual term. For example, $t_4 = 7$ tells us that the term in the fourth position is 7. We may need to talk about more than one sequence at a time. We can use different letters for these. Hence, we can use $t_1, t_2, t_3, \ldots$ for one sequence, $s_1, s_2, s_3, \ldots$ for another one, $u_1, u_2, u_3, \ldots$ for a third sequence, and so on.

Exercise: Can you write t_5, t_6, t_7 and t_8 for the sequence of triangular numbers?

There can be many kinds of sequences. For example, the terms could be fractions or negative integers. The ones we have discussed so far have terms that are increasing. But there could be sequences such as $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ where $t_1 = 1$ and the following terms are unit fractions occurring in decreasing order. We can also have sequences such as –7, –3, 1, 5, 9, ... where $s_1 = -7$, and consecutive terms have a difference of 4. Note that when we use the notation t_n for describing a term, n is always a non-negative integer but the term itself can be negative, or in fact any real number.

Think and Reflect

Can you think of any other kinds of sequences? List out five different types of sequences and discuss their properties with your friends.

8.2 Explicit Rule for a Sequence

Using the notation t_n (or s_n or u_n) we can write an explicit rule for the term in the n^{th} position of a sequence, that is, the n^{th} term. An **explicit formula** uses the term's position number, n, to calculate its value.

Example 1: Consider the expression $u_n = 2n - 1$. This states that the n^{th} term of the sequence is given by the rule $2n - 1$. When we substitute 1, 2, 3, ... for n in the expression $2n - 1$, we get $u_1 = 2 \times 1 - 1 = 1$, $u_2 = 2 \times 2 - 1 = 3$, $u_3 = 2 \times 3 - 1 = 5$, etc.

Thus, $u_n = 2n - 1$ is the explicit rule for the n^{th} term of the sequence of odd numbers.

Think and Reflect

Why is it useful to have an explicit formula for the n^{th} term of a sequence?

Using an explicit formula, we can find the 20th term, the 53rd term, the 300th term or any term of the sequence directly by just substituting the appropriate value of n. If we have an explicit formula, we can find the value of a term without having to know the value of previous terms!

Exercise: Using the explicit rule $u_n = 2n - 1$, find the 53rd term, the 108th term, and the 1170th term of the odd number sequence.

The explicit rule is useful in other ways too. We can use it to check if a certain number is a term of a sequence and also find the position of the term. For example, consider the odd number 137. To find which position it occupies in the odd number sequence, we need to solve the equation $u_n = 137$. This means $2n - 1 = 137$ or $n = 69$. Thus 137 is the 69th term of the odd number sequence.

Example 2: As another example, consider the sequence that is generated by the explicit formula $s_n = 5n - 2$. Can you write the first 6 terms of this sequence? What is the 100th term? The 1000th term?

Let us check if the numbers 308 and 473 are terms of this sequence.

We solve $s_n = 308$, that is, $5n - 2 = 308$. This leads to $5n = 310$ or $n = 62$. Thus 308 is the 62nd term of the sequence. Similarly solving $5n - 2 = 471$ leads to $5n = 473$, $n = 94.6$. Since 94.6 is not a natural number, we can conclude that 471 is not a term of the sequence. Can you explain why we need n to be a natural number?

Think and Reflect

Can you find the rule describing the n^{th} term of the sequence of square numbers?

Here is the sequence of the first ten prime numbers: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29. Do you see any pattern in this sequence? Can you think of a rule that can predict the next few prime numbers?

Exercise: Consider the expression $t_n = 3n - 7$.

(i) Find its first, second, third, 12th, 18th and 50th terms.

(ii) Which term of the sequence is 332?

(iii) Is 557 a term of this sequence? Why or why not?

8.3 Recursive Rule for a Sequence

So far, we have used a formula in terms of n for the n^{th} term as the explicit rule for a sequence. There is another way of writing the rule for a sequence. Consider the sequence 1, 4, 7, 10, 13, … . The n^{th} term is $t_n = 3n - 2$ (verify this for yourself). Note that each term is 3 more than its previous term. Thus, $t_2 = t_1 + 3$, $t_3 = t_2 + 3$, $t_4 = t_3 + 3$ and so on. In general, we can say $t_n = t_{n-1} + 3$ for $n \geq 2$. Or we can describe the sequence as $t_1 = 1$, $t_n = t_{n-1} + 3$, where n can take the values 2, 3, 4, … This way of describing a sequence by relating terms to previous terms is known as a **recursive rule** or **recursive formula**. If you know earlier terms of the sequence, then you can find the next terms using the rule. Note that the earlier terms need to be known to us to use the recursive rule to find the next terms.

Example 3: Find the first four terms of the sequence given by the recursive rule

$u_1 = 1$, $u_n = 2u_{n-1} + 3$ for $n \geq 2$. Is 133 a term of this sequence?

We successively insert the values of n as 2, 3, 4, etc., and compute the values of each term.

$$u_2 = 2u_1 + 3 = 2 \times 1 + 3 = 5$$
$$u_3 = 2u_2 + 3 = 2 \times 5 + 3 = 13$$
$$u_4 = 2u_3 + 3 = 2 \times 13 + 3 = 29$$

Thus, the first four terms of the sequence are 1, 5, 13, 29. Calculating subsequent terms, we can check if 133 is a term of this sequence.

Example 4: Find the first four terms of the sequence given by the recursive rule $s_1 = 3$, $s_n = s_{n-1}(s_{n-1} - 1)$ for $n \geq 2$.

Substitute the values of n as 2, 3, 4, etc., and compute the values of each term as follows.

$$s_2 = s_1(s_1 - 1) = 3 \times (3 - 1) = 3 \times 2 = 6$$
$$s_3 = s_2(s_2 - 1) = 6 \times (6 - 1) = 6 \times 5 = 30$$
$$s_4 = s_3(s_3 - 1) = 30 \times (30 - 1) = 30 \times 29 = 870$$

Thus, the first four terms of the sequence are 3, 6, 30, 870.

Virahānka–Fibonacci sequence

A recursive rule or formula does not only have to involve the previous term — it could involve the previous two or more terms.

The most famous example of such a sequence is V_1, V_2, V_3, ... where $V_1 = 1$, $V_2 = 2$, and $V_n = V_{n-1} + V_{n-2}$ for $n \geq 3$. We compute:

$$V_3 = V_2 + V_1 = 2 + 1 = 3$$
$$V_4 = V_3 + V_2 = 3 + 2 = 5$$
$$V_5 = V_4 + V_3 = 5 + 3 = 8$$

So we get the sequence

$$1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

where each term is obtained by adding the previous two.

Can you write the next two terms of this sequence?

Do you recognise this sequence from previous grades? That's right, it is the Virahānka–Fibonacci sequence! It was first written down explicitly and studied by Virahānka in his work *Vṛttajātisamuchaya* in the 7th century CE. He discovered it in the context of Prakrit meter and poetry! It was further studied by the linguist-mathematicians Gopāla (c. 1135 CE) and Hemachandra (c. 1150 CE). The sequence was later also studied by the Italian mathematician Fibonacci (c. 1200 CE).

The Virahānka sequence plays an important role throughout mathematics and science. We will encounter it again many times in later grades.

Exercise Set 8.1

1. Find the first five terms of the sequence in which the n^{th} term is given by (i) $t_n = 3n - 4$, (ii) $t_n = 2 - 5n$, and (iii) $t_n = n^2 - 2n + 3$ for $n \geq 1$.
2. Find the 10^{th} and 15^{th} terms of the sequence $t_n = 5n - 3$ for $n \geq 1$.
3. Determine whether 97 and 172 are terms of the sequence $t_n = 5n - 3$ for $n \geq 1$.
4. Which term of the sequence $t_n = 5n - 3$ for $n \geq 1$ is 607?
5. A sequence is given by the recursive rule $t_1 = -5$, $t_{n+1} = t_n + 3$ for $n \geq 1$. Find the first five terms of the sequence. Is 52 a term of this sequence? If so, which term is it?

6. Let $T_1 = 1$, $T_2 = 2$, $T_3 = 4$, and $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for $n \geq 4$. Find T_4, T_5, T_6, T_7, and T_8.

8.4 Arithmetic Progressions

So far we have learnt that a sequence is an ordered list of numbers that may follow a particular rule. However, there may be sequences, such as the sequence of prime numbers, where there is no clear regularity in the rule.

In this section, we explore a special kind of sequence called an **arithmetic progression**.

Let us look at the growing pattern of squares given in Fig. 8.3. The first four stages of the pattern are shown.

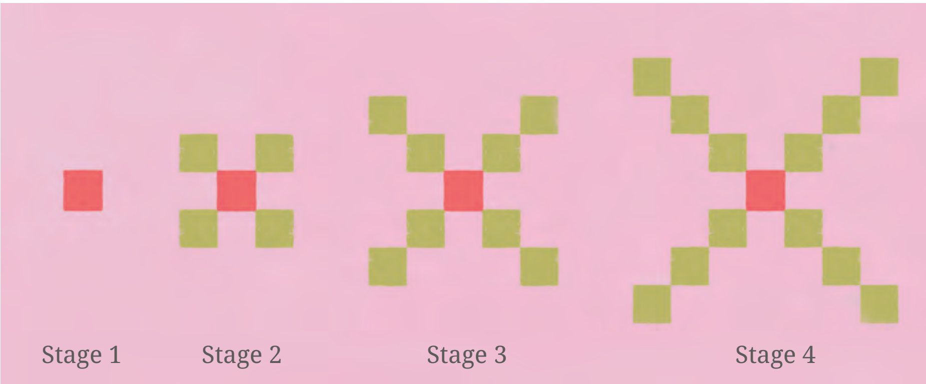


Fig. 8.3: Growing pattern of squares

If we count the number of tiny squares at each stage, we get sequence 1, 5, 9, 13.

Think and Reflect

Can you predict the number of squares in Stages 5 and 6 of the sequence? In Stages 10, 11 and 12? In Stage 20? At any stage?

We observe that at each stage, 4 squares get added to the corners of the pattern in the earlier stage. The number of squares at successive stages can be written as

$$1, 1 + 4, 1 + 4 + 4, 1 + 4 + 4 + 4, \ldots$$

This may be rewritten as

$$1, 1 + 1 \times 4, 1 + 2 \times 4, 1 + 3 \times 4, \ldots$$

If we treat these expressions as the terms of a sequence, we get

$$t_1 = 1, t_2 = 1 + 1 \times 4, t_3 = 1 + 2 \times 4, t_4 = 1 + 3 \times 4$$

and so on. We may therefore write the n^{th} term as $t_n = 1 + (n - 1) \times 4$, which simplifies to $t_n = 4n - 3$.

The first six terms of the sequence representing the number of squares in Fig. 8.3 are 1, 5, 9, 13, 17, 21. We note that the difference between successive terms is the constant value 4. Such sequences, in which the difference between consecutive terms is constant, are known as arithmetic progressions. We will refer to them as APs.

Think and Reflect

Consider all the sequences we have discussed so far in this chapter. Which ones are arithmetic progressions and which ones are not? Can you justify your claim?

Note that in the n^{th} term $t_n = 1 + (n - 1) \times 4$ of the sequence above, 1 is the first term of the sequence and 4 is the 'common difference'. Similarly, the n^{th} term of the sequence 1, 4, 7, 10, ... is $t_n = 1 + (n - 1) \times 3$, where 1 is the first term and 3 is the common difference. In the sequence 11, 7, 3, –1, –5, ... the numbers decrease by 4. This is also an arithmetic progression, where the first term is 11 and the common difference is –4.

In general, an arithmetic progression (AP) can be described as

$$a, a + d, a + 2d, a + 3d, \ldots, a + (n - 1) \times d,$$

where 'a' is the **first term** and 'd' is the **common difference**. Thus $t_n = a + (n - 1) \times d$ is an expression for the n^{th} term of any arithmetic progression, for some fixed values of a and d.

8.4.1 Visualising an AP

Let us return to the growing pattern of squares in Fig. 8.3 and prepare a table that shows the number of tiny squares at each stage.

Stage Number	1	2	3	4	5	...	n
Number of squares	1	5	9	13	17	...	$4n - 3$

Let us form a pair of numbers (x, y) using the information in the table above where x represents the stage number and y the corresponding number of squares. When we plot the ordered pairs emerging from the table, that is, (1, 1), (2, 5), (3, 9), (4, 13), (5, 17), we observe that they lie on a straight line! This is shown in Fig. 8.4.

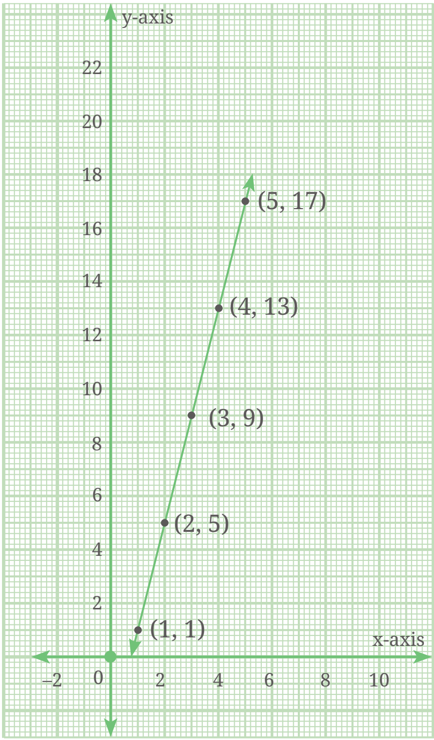


Fig. 8.4: Growing pattern of squares represented by a linear pattern

Exercise: Verify that the following sequences are arithmetic progressions and write their n^{th} terms. What do you observe when you plot the ordered pairs emerging from them?

(i) 2, 5, 8, 11, … (ii) –5, –1, 3, 7, …

Exercise: Using the formula $t_n = a + (n - 1) \times d$, find the n^{th} term of the following arithmetic progressions.

(i) $\frac{1}{2}, \frac{5}{2}, \frac{9}{2}, \frac{13}{2}, \ldots$ (ii) 1.5, 3.5, 5.5, 7.5, …

Note that $t_n = a + (n - 1) \times d$ is the explicit rule for finding the n^{th}

term of an AP. Can we also find the recursive rule? Yes. It is $t_1 = a$, $t_n = t_{n-1} + d$ for $n \geq 2$.

Consider once again the AP: 1, 5, 9, 13, 17, ... Since every term is 4 more than the previous term, another way of writing this sequence is $t_1 = 1$, $t_n = t_{n-1} + 4$ for $n \geq 2$. Verify for yourself that this recursive rule leads to the terms of the sequence.

Exercise: Find recursive rules for the APs in the previous exercises.

Example 5: A person books a taxi to travel in the city. The taxi company charges a fixed booking fee of ₹200 plus ₹40 per kilometre travelled. Let us write the sequence representing the total fare after travelling 1 km, 2 km, 3 km, and so on. If the person travels 10 km, what will be the total fare?

Since the fixed amount is ₹200, after 1 km the fare will be ₹200 + ₹40 = ₹240, after 2 km it will be ₹200 + ₹80 = ₹280 and after 3 km it will be ₹200 + ₹120 = ₹320. The sequence 240, 280, 320, ... is an AP with first term 240 and common difference 40. The n^{th} term of the sequence is $240 + (n - 1) \times 40 = 240 + 40n - 40 = 200 + 40n$ where n represents the distance travelled in km.

8.5 Sum of the First *n* Natural Numbers

In this section, we derive a simple yet important rule for the sum of the first n natural numbers. To begin with, can you find the sum of the first ten natural numbers without actually adding all of them?

Let us try a unique approach. Let S denote the sum of the first ten natural numbers. Thus S = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10. We can also write it as S = 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1. If we place these two equations, one below the other, we notice something interesting.

$$S = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$$
$$S = 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1$$

Each pair of corresponding numbers, 1 + 10, 2 + 9, 3 + 8, ... , 10 + 1, in the two equations add up to 11.

Adding the two equations for S, we get

$$2S = 11 + 11 + 11 + \cdots + 11 \text{ (that is, 11 added 10 times).}$$

This leads to 2S = 110 or S = 55. The sum of the first 10 natural numbers is indeed 55.

Think and Reflect

Can the same approach be used to find the sum of 1 + 2 + 3 + ⋯ + 100?

This method of writing the sum from 1 to any number, reversing the sum and then adding the two expressions can be used to arrive at the formula for the sum of the first n natural numbers for any value of n.

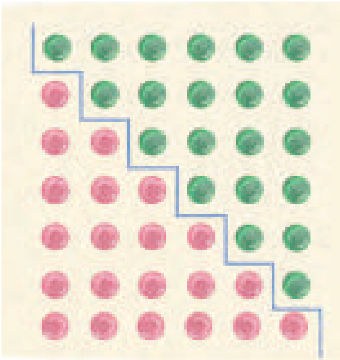
Let S = 1 + 2 + ⋯ + n.

Then S = $n + (n - 1) + \cdots + 1$.

So, 2S = $n(n + 1)$. We conclude S = $\frac{n(n+1)}{2}$.

We can also think of this argument pictorially.

The diagram in Fig. 8.5 represents the method to find the sum 1 + 2 + 3 + 4 + 5 + 6. Note that the figure comprises two sets of circles that represent the sum 1 + 2 + 3 + 4 + 5 + 6 above and below of the zigzag partition. The total number of circles forms a 7 × 6 rectangular array.



2 × (1 + 2 + 3 + 4 + 5 + 6) = 7 × 6

Fig. 8.5

Thus, Fig. 8.5 represents the fact 2 × (1 + 2 + 3 + 4 + 5 + 6) = 7 × 6.

This can similarly be extended to find the sum

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10.$$

Namely, 2 × (1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10) = 10 × 11.

In general, $2 \times (1 + 2 + 3 + 4 + \cdots + n) = n \times (n + 1)$.

Therefore, $1 + 2 + 3 + 4 + \cdots + n = \frac{n(n+1)}{2}$.

If we use the notation S_n to represent the sum of the first n natural numbers, then $S_n = \frac{n(n+1)}{2}$.

The first known written mention of this result can be found in Āryabhaṭa's *Āryabhaṭīya*, Chapter 2, Verse 19. The verse provides two ways to calculate the sum, with the second part describing that the sum can be calculated by taking the sum of the first and last terms, divided by two (the average), and multiplied by the number of terms.

Think and Reflect

Can you use this formula to find S_{20}, S_{50} or S_{1000} ?

Also, this formula can be used to find the sum of consecutive numbers such as 25 + 26 + 27 + ⋯ + 58.

Note that 25 + 26 + 27 + ⋯ + 58 = (1 + 2 + 3 + ⋯ + 58) – (1 + 2 + 3 + ⋯ +24)

$$= S_{58} - S_{24} = \frac{58 \times 59}{2} - \frac{24 \times 25}{2} = 29 \times 59 - 12 \times 25 = 1711 - 300 = 1411.$$

Think and Reflect

Let us revisit the sequence t_n of triangular numbers 1, 3, 6, 10, 15, ... shown in Fig. 8.1. Note that the n^{th} term of this sequence is the sum of the first n natural numbers. Thus $t_n = \frac{n(n+1)}{2}$.

Can you use this to find the 10th, 17th and 80th triangular numbers?

Exercise Set 8.2

1. Find the 10th and 26th terms of the AP: 3, 8, 13, 18, ...
2. Which term of the AP : 21, 18, 15, ... is – 81? Also, is 0 a term of this AP? Give reasons for your answer.
3. Find the n^{th} term of the AP: 11, 8, 5, 2 ... Write the recursive rule for this AP.
4. An AP consists of 50 terms in which the 3rd term is 12 and the last term is 106. Find the 29th term.
   (**Hint:** If '*a*' is the first term and 'd' the common difference, then we arrive at the equations $a + 2d = 12$ and $a + 49d = 106$. Solve this pair of linear equations for '*a*' and '*d*'.)

5. How many 2-digit numbers are divisible by 3? What is the sum of all these 2-digit numbers?
6. Harish started work at an annual salary of ₹5,00,000 and received an increment of ₹20,000 each year. After how many years did his income reach ₹7,00,000?
7. A child arranges marbles in rows so that the first row has 1 marble, the second has 2 marbles, the third has 3, and so on up to 25 rows. How many marbles does the child use in all?

8.6 Geometric Progressions

In this section, we explore yet another special kind of sequence called a **geometric progression (GP)**. Consider the growing pattern of squares shown in Fig. 8.6. The first four stages of the pattern are shown.

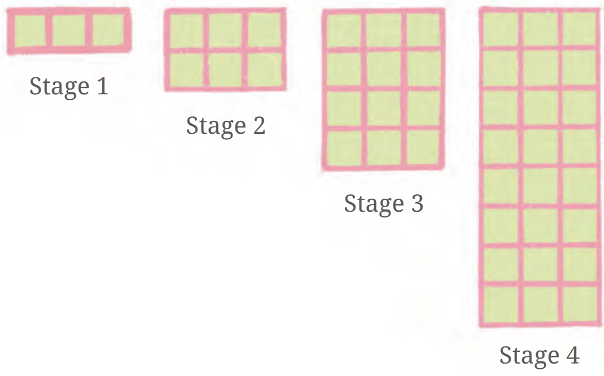


Fig. 8.6: A growing pattern of squares

If we count the total number of green squares in the four stages of the pattern, we get the sequence 3, 6, 12, 24.

Think and Reflect

Can you predict the number of squares in Stages 5 and 6 of the pattern? In Stages 10, 11 and 12? In Stage 20? At any stage? How is this different from the growing pattern in Fig. 8.3?

We observe that at each stage, the number of green squares is doubled or is twice the number of squares in the previous stage. The number of squares at successive stages is 3, 3 + 3 = 6, 6 + 6 = 12, 12 + 12 = 24, ...

These may be rewritten as

$$3, 3 \times 2, 3 \times 4, 3 \times 8, \ldots \text{ or as } 3, 3 \times 2, 3 \times 2^2, 3 \times 2^3, \ldots$$

If we treat these expressions as the terms of a sequence, we get

$$t_1 = 3, t_2 = 3 \times 2, t_3 = 3 \times 2^2, t_4 = 3 \times 2^3$$

and so on. We can write the n^{th} term as $t_n = 3 \times 2^{n-1}$. As a recursive formula, we can write $t_1 = 3$ and $t_n = 2t_{n-1}$ for $n \geq 2$.

The first six terms of the sequence are 3, 6, 12, 24, 48, 96. Each term of the sequence is obtained by multiplying the previous term by 2. This **constant multiplier** is also known as the **common ratio** of the sequence. We see that the ratios of consecutive pairs of terms are all the same.

$$\frac{6}{3} = \frac{12}{6} = \frac{24}{12} = \frac{48}{24} = \frac{96}{48} = 2.$$

Such a sequence with a common ratio is known as a **Geometric Progression or GP.** In general, a GP may be described in the form of a sequence as $a, ar, ar^2, ar^3, \ldots, ar^{n-1}$, where '$a$' is the **first term**, 'r' is the **common ratio** and $t_n = ar^{n-1}$ is the **n^{th} term**.

Example 6: Is 1, 2, 4, 8, 16, … a geometric progression? If so, what is the common ratio?

Example 7: Is 1, 3, 9, 27, 81, … a geometric progression? If so, what is the common ratio?

Example 8: Is 1, –1, 1, –1, 1, … a geometric progression? If so, what is the common ratio?

Example 9: Check whether the sequence $5, \frac{15}{4}, \frac{45}{16}, \frac{135}{64}, \ldots$ is a geometric progression and find its n^{th} term.

In the given sequence $t_1 = 5, t_2 = \frac{15}{4}, t_3 = \frac{45}{16}, t_4 = \frac{135}{64}$

Let us evaluate the ratios of consecutive pairs of terms.

$$\frac{t_2}{t_1} = \frac{15}{4} \div 5 = \frac{15}{4} \times \frac{1}{5} = \frac{3}{4}.$$

$$\frac{t_3}{t_2} = \frac{45}{16} \div \frac{15}{4} = \frac{45}{16} \times \frac{4}{15} = \frac{3}{4}.$$

$$\frac{t_4}{t_3} = \frac{135}{64} \div \frac{45}{16} = \frac{135}{64} \times \frac{16}{45} = \frac{3}{4}.$$

The ratio of consecutive pairs of terms is a constant $\frac{3}{4}$. Hence the given sequence is a GP with $a = 5$ and $r = \frac{3}{4}$. The n^{th} term $= a \times r^{(n-1)}$ $= 5 \times \left(\frac{3}{4}\right)^{n-1}$. Substitute $n = 1, 2, 3, \ldots$ in this expression to check if you get the terms of the given sequence.

Exercise: Check whether the following sequences are geometric progressions and find their n^{th} terms.

(i) 2, 10, 50, 250, …

(ii) $4, \frac{8}{3}, \frac{16}{9}, \frac{32}{27}, \ldots$

(iii) $3, \frac{-3}{2}, \frac{3}{4}, \frac{-3}{8}, \ldots$

Exercise: Can you find a recursive rule for the formula $t_n = 3 \times 10^{n-1}$ that generates the geometric progression 3, 30, 300, 3000, … ?

8.6.1 Fun with Fractals

Recall from Grade 8 another interesting pattern as shown in Fig. 8.7. Stage 0 represents a piece of paper cut in the shape of an equilateral triangle. Let us join the midpoints of the three sides of the triangle leading to four smaller equilateral triangles. Now remove the central triangle. You will get the figure in Stage 1 with a triangular hole in the centre. Repeat the process on all three black triangles in Stage 1. This will lead to the figure in Stage 2. In a similar manner, we arrive at Stage 3 from Stage 2. The process can be continued forever! This fascinating pattern is known as the **Sierpiński triangle**. This is actually a fractal. We will learn more about fractals in a later grade.

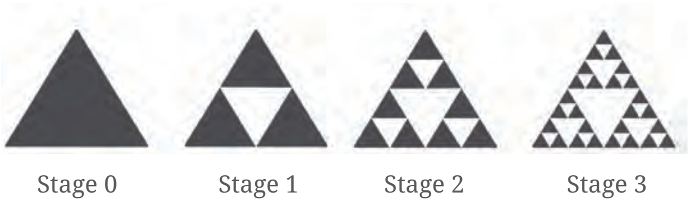


Fig. 8.7

Wacław Sierpiński (1882 – 1969) was a Polish mathematician known for his numerous contributions to mathematics, including the Sierpiński gasket, one of the earliest and most famous examples of a fractal.

Think and Reflect

Observe the Sierpiński triangle and try to answer the following questions

(a) How many black triangles are there in Stages 0 to 3 of Fig. 8.7?

(b) Can you predict the number of black triangles at Stages 4 and 5?

(c) Can you find a rule for the number of black triangles at the n^{th} stage?

(d) Suppose the area of the triangle (that is, the black region) in Stage 0 is 1 square unit. What is the area of the black region in Stages 1, 2 and 3? What will be the area of the black region in Stages 4 and 5? Find a rule for the area of the black region at the n^{th} stage. What happens to this area as n, the number of stages, goes on increasing?

We observe that the number of black triangles is 1, 3, 9 and 27 in Stages 0, 1, 2 and 3 respectively. In fact, in every stage, each black triangle is replaced with three smaller triangles at the next stage. Thus, the number of black triangles at every stage is three times the number at the previous stage.

At Stages 4 and 5 the number of black triangles will be 81 and 243.

Note that the sequence 1, 3, 9, 27, 81, 243, ... is a GP since every successive term of the sequence can be obtained by multiplying the previous term by 3. Also, all the terms of this sequence are powers of 3: $1 = 3^0$, $3 = 3^1$, $9 = 3^2$ and so on. Continuing in this manner we observe that the exponent of 3 matches the stage number, as shown in Table 1. Thus, the number of black triangles at the n^{th} stage of the Sierpiński triangle is given by 3^n. The number of black triangles increases very quickly as the stage numbers increase. Can you explain why?

What about the area of the black region? In Stage 1, the equilateral triangle at Stage 0 is divided into 4 equal parts and the central part is removed. This means that, if the black region at Stage 0 is 1 square unit, then then the black region at Stage 1 is $\frac{3}{4}$ square units. This process is repeated on Stage 1 to arrive at Stage 2. Hence, the black region at Stage 2 will be $\frac{3}{4}$ of the black region of Stage 1, which is equal to $\frac{3}{4} \times \frac{3}{4} = \left(\frac{3}{4}\right)^2$. Can you explain why the area of the black region at Stage n will be $\left(\frac{3}{4}\right)^n$?

The area of the black region at each stage also leads to a geometric progression where every successive term of the sequence is obtained by multiplying the previous term by $\frac{3}{4}$. Thus, while the number of black triangles increases rapidly, the total area of the triangles decreases.

Table 1

Stage (n)	0	1	2	3	4	5	...	n
Number of black triangles (t_n)	$1 = 3^0$	$3 = 3^1$	$9 = 3^2$	$27 = 3^3$	$81 = 3^4$	$243 = 3^5$	...	3^n
Shaded area (s_n)	1	$\frac{3}{4}$	$\left(\frac{3}{4}\right)^2$	$\left(\frac{3}{4}\right)^3$	$\left(\frac{3}{4}\right)^4$	$\left(\frac{3}{4}\right)^5$	...	$\left(\frac{3}{4}\right)^n$

The explicit formula for the number of black triangles and the area of the black region at any stage is given by $t_n = 3^n$ and $s_n = \left(\frac{3}{4}\right)^n$ respectively. The recursive rules for the same are

$$t_1 = 1,\ t_n = 3 \times t_{n-1} \text{ for } n \geq 2 \text{ and } s_1 = 1,\ s_n = \frac{3}{4} \times s_{n-1} \text{ for } n \geq 2.$$

Fractals are shapes or patterns that repeat themselves at different scales. This means that if you zoom in on a small part of a fractal, it looks similar to the whole! Fractals are found throughout nature—like in the branching of trees, in vegetables, such as cauliflower or broccoli, in the shape of snowflakes, or in the patterns of coastlines. They can be created using simple rules but can form very complex and beautiful designs. Fractals help us understand patterns in nature, and also lead us to important concepts in mathematics.

Fig. 8.8

8.6.2 Visualising a GP

Let us revisit the growing pattern of squares in Fig. 8.6 and prepare a table that shows the number of green squares at each stage.

Stage number	1	2	3	4	5	...	n
Number of squares	3	6	12	24	48	...	$3 \times 2^{n-1}$

Let us consider the pairs of numbers (x, y) where x represents the stage number and y the corresponding number of squares. When we plot the ordered pairs emerging from the above table, that is, (1, 3), (2, 6), (3, 12), (4, 24), (5, 48), we observe that they do not lie on a straight line! This is shown in Fig. 8.9.

Similarly, when we plot the pairs of numbers (x, y) where x represents stage number and y the number of black triangles in the Sierpiński triangle we get the graph in Fig. 8.10 A. Fig. 8.10 B shows the graph where stage numbers are represented on the x-axis and the area of the black region of the stages on the y-axis. The graphs tell us that as the stage numbers increase, the number of shaded triangles increases very quickly whereas the area of black region diminishes, getting closer and closer to 0.

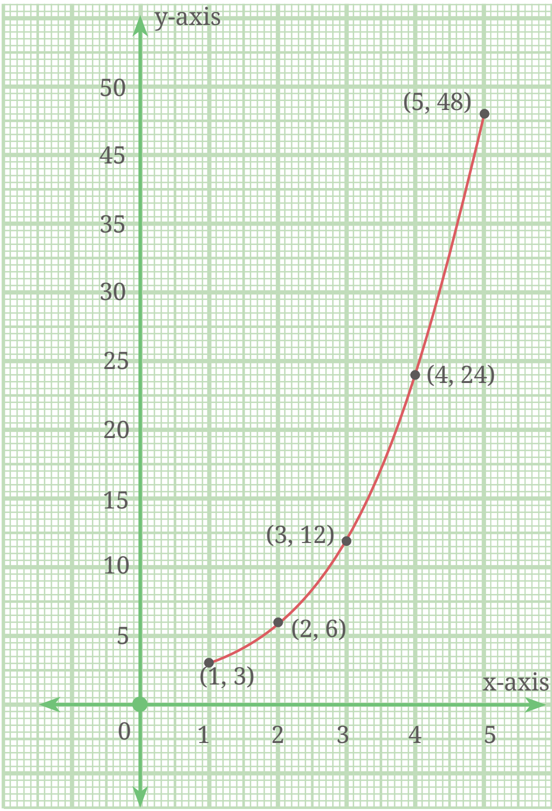
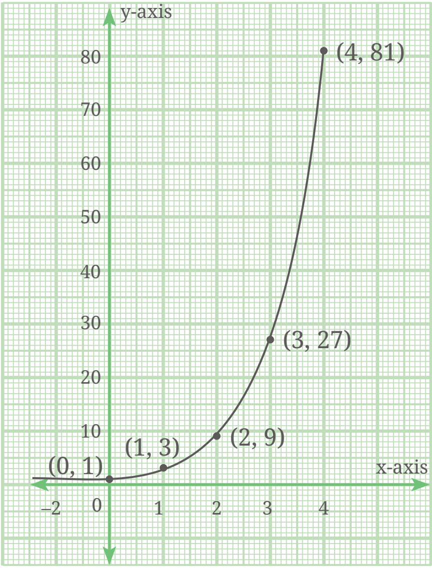
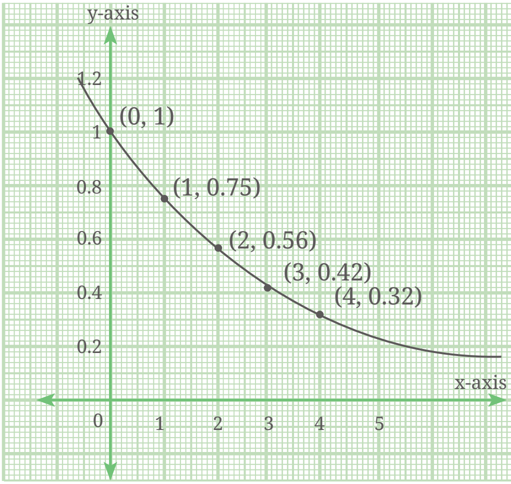


Fig. 8.9: Points emerging from a GP do not lie on a straight line



(A)



(B)

Fig. 8.10: Points emerging from a GP arising out of the stages of the Sierpiński Triangle

Here is another example in which a GP arises.

Example 10: A ball is dropped from a height of 24 feet above the ground. Each time the ball bounces up to $\left(\frac{3}{4}\right)^{\text{th}}$ of its previous height.

(a) Can you write the sequence of numbers obtained from the heights attained by the ball in five successive bounces?

(b) How many bounces are required for the ball to remain below a height of $\frac{1}{6}$ of the original height from which it was dropped?

The sequence of maximum heights attained by the ball after each bounce is:

First bounce: $24 \times 0.75 = 18$ feet

Second bounce: $18 \times 0.75 = 13.5$ feet

Third bounce: $13.5 \times 0.75 = 10.125$ feet

Fourth bounce: $10.125 \times 0.75 = 7.59375$ feet

Fifth bounce: $7.59375 \times 0.75 = 5.695$ feet

Sixth bounce: $5.695 \times 0.75 = 4.27125$ feet

Seventh bounce: $4.27125 \times 0.75 = 3.2034375$ feet

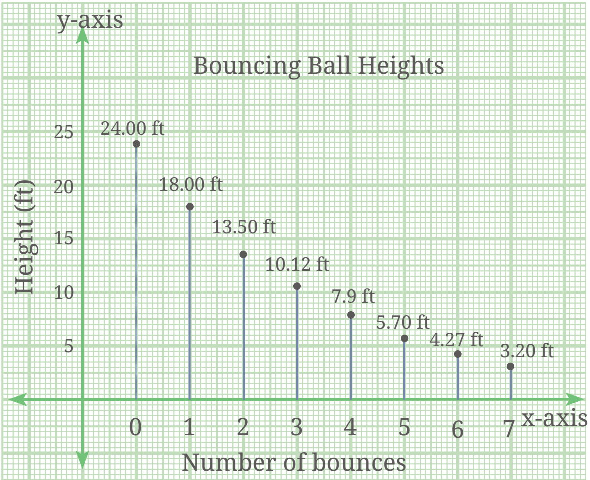


Fig. 8.11

The sequence of heights forms a GP with a = 18 and r = 0.75 (or $\frac{3}{4}$), that is, 18, 13.5, 10.125, 7.594, 5.695, 4.27125, 3.2034375, ... We see that after the seventh bounce the ball remains below $\frac{1}{6}$ of the height at which it started.

Exercise Set 8.3

1. Find the 12th term of a GP with common ratio 2, whose 8th term is 192.
2. Find the 10th and n^{th} terms of the GP: 5, 25, 125, ...

*3. A sequence is given by the recursive rule $t_1 = 2$, $t_{n+1} = 3t_n - 2$ for $n \geq 1$. Which term of the sequence is 730?

4. Which term of the GP: 2, 6, 18, ... is 4374? Write the explicit formula as well as the recursive formula for the n^{th} term.
5. A ball is dropped from a height of 80 metres. After hitting the ground, it bounces back to 60% of the height from which it fell. It continues bouncing in this way—each time rising to 60% of the previous height.
   (i) What height does the ball reach after the 5th bounce?
   (ii) What is the total vertical distance the ball has travelled by the time it hits the ground for the 6th time?

6. Which term of the sequence $2, 2\sqrt{2}, 4, \ldots$ is 128?
7. Fig. 8.12 shows Stages 0 to 3 of the Sierpiński square carpet. Stage 0 of this fractal is a square sheet of paper. To construct Stage 1, each side of the square is trisected and the points of trisection of opposite sides are joined to obtain nine smaller squares. The centre square is then removed and the 8 smaller squares are retained, leaving a square hole in the centre. The same process is repeated on the eight smaller shaded squares to obtain Stage 2 and so on.

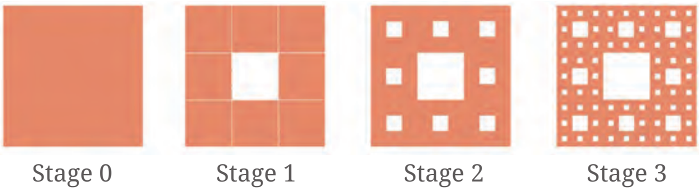


Fig. 8.12: Stages 0, 1, 2 and 3 of the Sierpiński square carpet

Look at Fig. 8.12 and try to answer the following questions.

(i) How many red squares are there in Stages 0 to 3?

(ii) Can you predict the number of red squares in Stages 4 and 5?

(iii) Can you find a rule for the number of red squares at the n^{th} stage? Write the explicit formula as well as the recursive formula for the number of red squares at any stage.

(iv) Suppose the area of the square in Stage 0 is 1 square unit. What is the area of the red region in Stages 1, 2 and 3? What will be the area of the red region in Stages 4 and 5? Find the explicit as well as the recursive formula for the area of the red region at the n^{th} stage. What happens to this area as n, the number of stages, goes on increasing?

End-of-Chapter Exercises

1. Find the 31^{st} term of an AP whose 11^{th} term is 38 and 16^{th} term is 73.
2. Determine the AP whose third term is 16 and whose 7^{th} term exceeds the 5^{th} term by 12.

*3. How many three-digit numbers are divisible by 7? (**Hint:** All three-digit numbers divisible by 7 form an AP. Find the smallest and largest such three-digit numbers.)

*4. How many multiples of 4 lie between 10 and 250? (**Hint:** All multiples of 4 form an AP. Find the smallest and largest multiples of 4 between 10 and 250.)

*5. Find a GP for which the sum of the first two terms is –4 and the fifth term is 4 times the third term.

*6. Find all possible ways of expressing 100 as the sum of consecutive natural numbers.

*7. The number of bacteria in a certain culture doubles every hour. If there were 30 bacteria present in the culture originally, how many bacteria will be present at the end of the 2nd hour, 4th hour and n^{th} hour?

*8. The sum of the 4th and 8th terms of an AP is 24 and the sum of the 6th and 10th terms is 44. Find the first three terms of the AP.

*9. Find the smallest value of n such that the sum of the first n natural numbers is greater than 1,000.

*10. Which term of the GP: 2, 8, 32, ... is 131072? Write the explicit formula as well as the recursive formula for the n^{th} term.

*11. The sum of the first three terms of a GP is $\frac{13}{12}$ and their product is –1. Find the common ratio and the terms.

*12. If the 4th, 10th and 16th terms of a GP are x, y and z respectively, prove that x, y, z are in GP.

*13. The sum of the first three terms of a geometric progression is 26, and the sum of their squares is 364. Find the terms of the GP.

*14. Suppose $P_1 = 1$, $P_2 = 2$ and for $n > 2$, $P_n = P_1 + P_2 + \cdots + P_{n-1} + 1$. Find the values of $P_1, P_2, \ldots, P_8$. Can you find a simpler recursive formula for P_n? Can you give an explicit formula?

*15. Suppose $W_1 = 1$, $W_2 = 2$ and for $n > 2$, $W_n = W_1 + W_2 + \cdots + W_{n-2} + 2$. Find the values of $W_1, W_2, \ldots, W_8$. Do you recognise this sequence?

CHAPTER SUMMARY

- A **sequence** is an ordered list of numbers. Each number in the list is called a term.
- A **general formula** or rule for a sequence is a rule that can be used to generate each term. An **explicit formula** is a rule that uses the term's position number, n, to calculate the term's value.
- A **recursive formula** is a rule that gives the value of a given term using the values of previous terms.
- The **triangular number sequence** is given by 1, 3, 6, 10, 15, … . Each term is the sum of the natural numbers up to the position of that term. The n^{th} term is given by $t_n = \frac{n(n+1)}{2}$ which is also the formula for the sum of the first n natural numbers.
- An **arithmetic progression** (AP) is a list of numbers in which each term after the first term is obtained by adding a fixed number d to the previous term. The fixed number d is called the **common difference**.
- The n^{th} term of an AP is given by $t_n = a + (n - 1)\, d$, where a is the first term and d is the common difference. The general form of an AP is $a, a + d, a + 2d, a + 3d, \ldots, a + (n - 1)\, d$.
- A **geometric progression** (GP) is a list of numbers in which each term after the first term is obtained by multiplying the previous term by a fixed number. This constant factor r is called the **common ratio**.
- The n^{th} term of a GP is given by $t_n = ar^{n-1}$, where 'a' is the first term and 'r' is the common ratio. The general form of a GP is $a, ar, ar^2, ar^3, \ldots, ar^{n-1}$.
- Many attributes of fractals, such as the Sierpiński triangle and the Sierpiński square carpet, lead to geometric progressions.